Combinatorial Mathematics

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Monday 18:30 – 20:20

Outline

- Basic counting principles
- Arrangements and Selections w/ or w/o Repetitions (P.11, P.21)
- Distributions (P.32)
- Binomial Identities & The Double-Counting Principle (P.38)
- Principle of Inclusion-Exclusion (1) (P.46)

Basic Counting

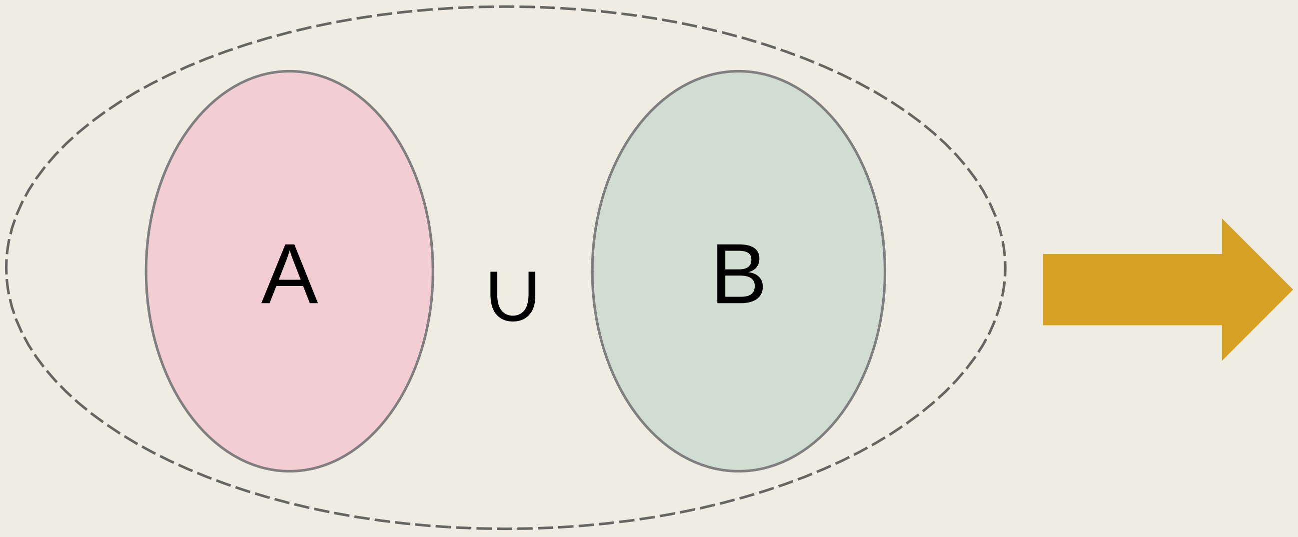
Counting - the oldest & most important combinatorial reasoning tool.

The Addition Principle

If there are r_1 different objects in the first set and
r_2 different objects in the second set, and

if the two sets are disjoint,

then the number of ways to select an object from the two sets is $r_1 + r_2$.



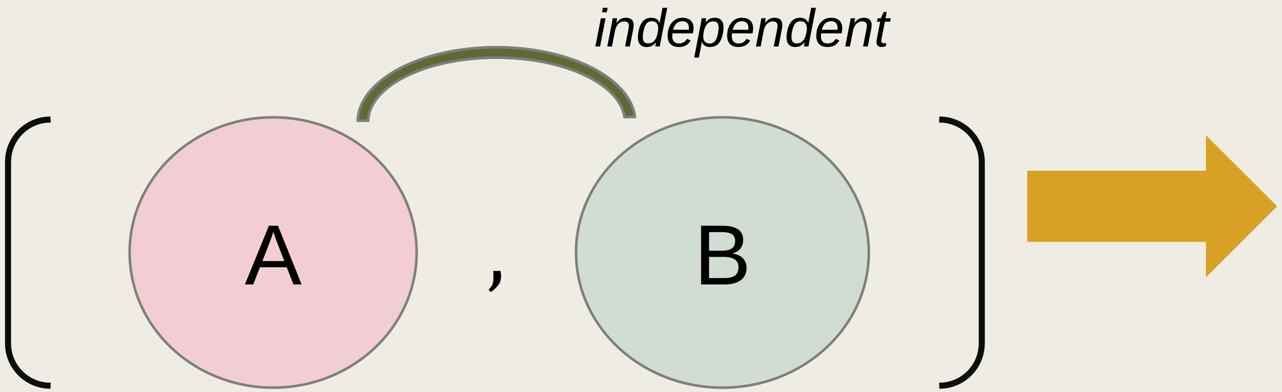
Number of outcomes

$$|A| + |B|$$

The Multiplication Principle

Suppose that a procedure can be broken into two ***successive (ordered)*** stages, with r_1 different outcomes in the first stage and r_2 different outcomes in the second stage.

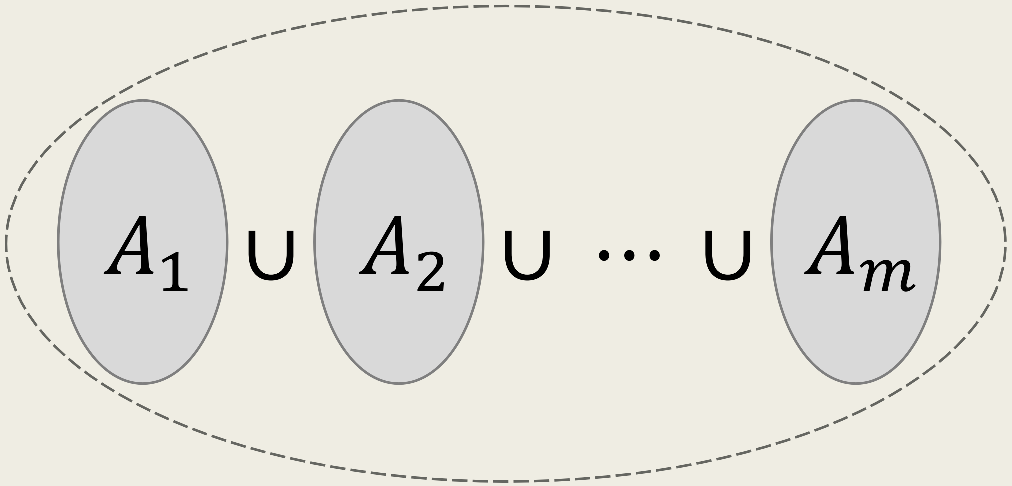
If the number of outcomes at the second stage is independent of the choices in the previous stage, and if the composite outcomes are all distinct, then the total number of different outcomes is $r_1 \times r_2$.



Number of outcomes

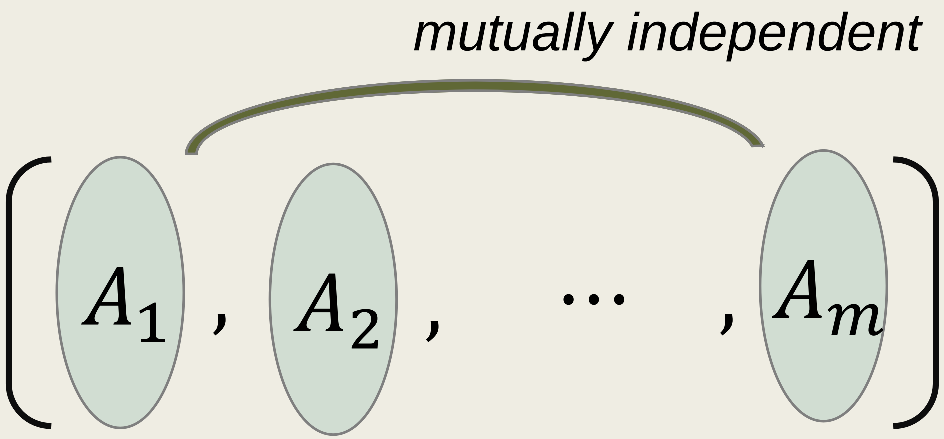
$$|A| \times |B|$$

The principles extends to multiple sets and stages.



Number of outcomes

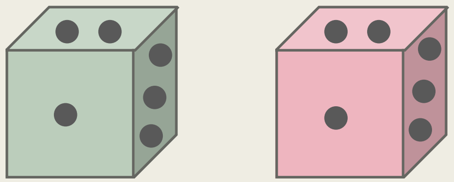
$$|A_1| + |A_2| + \cdots + |A_m|$$



Number of outcomes

$$|A_1| \times |A_2| \times \cdots \times |A_m|$$

Example 1. Rolling Dice



- Two dice are rolled, one green and one red.
  Each dice has faces numbered 1 through 6.

  a. How many different outcomes of this procedure are there?

  b. What is the probability that there are no doubles
     (not the same value on both dice) ?

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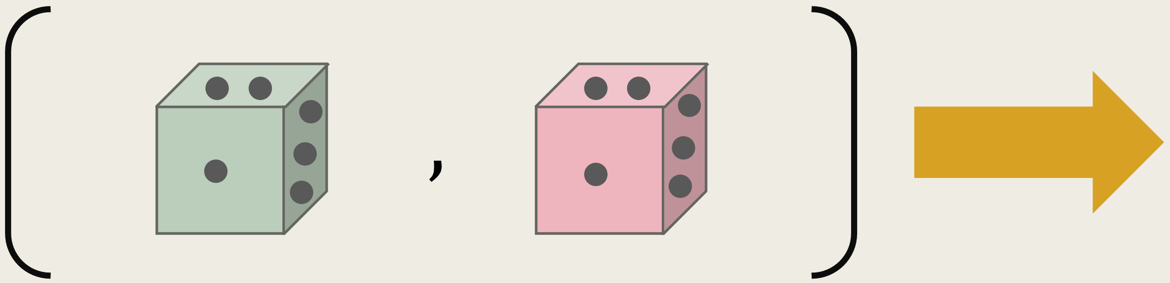


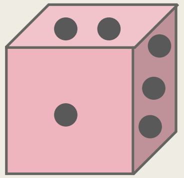
Number of outcomes

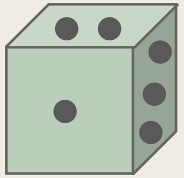
$$6 \times 6 = 36$$

Example 1. Rolling Dice

- Two dice are rolled, one green and one red.
  Each dice has faces numbered 1 through 6.

  b. What is the probability that there are no doubles (not the same value on both dice) ?





For <u>each value</u> of the first dice,
there are 5 different values for the second dice to form a non-double result.

In total, there are $5 + 5 + 5 + 5 + 5 + 5 = 30$ such outcomes.

The probability is $30/36 = 5/6$.

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- Two dice are rolled, one green and one red.
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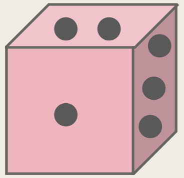

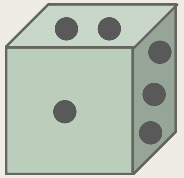

Alternatively, there are 6 double outcomes.

So, the probability is $1 - \frac{6}{36} = 5/6$.

Simple Arrangement & Selections

Counting when all the objects are distinct.

Permutation

A permutation of n distinct objects is an arrangement, or ordering, of the n objects.

An r-permutation of n distinct objects is an arrangement using r of the n objects.

- Let $P(n, r)$ denote the number of r-permutations of a set of n distinct objects.

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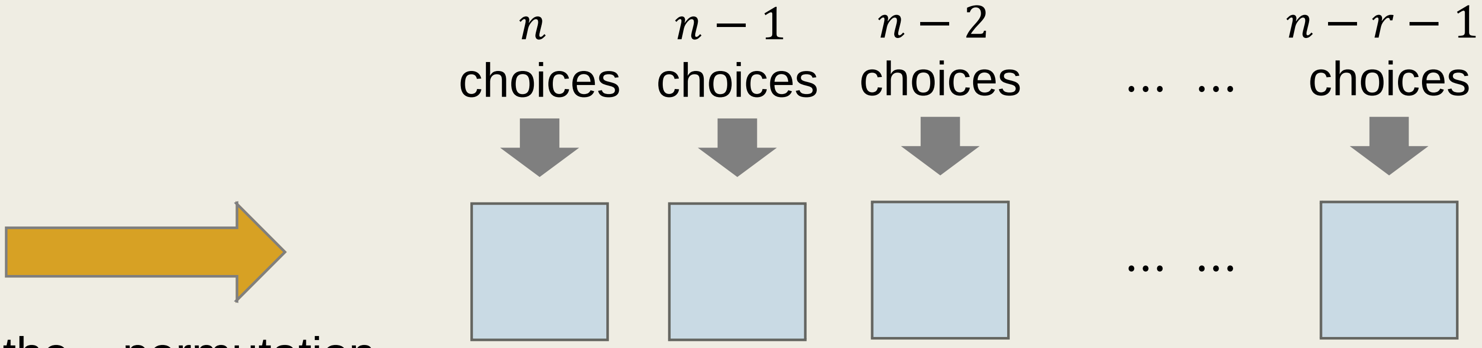
- Let $P(n,r)$ denote the number of r-permutations of a set of n distinct objects.



Decide the r-permutation (r-ordered pair) in this order.

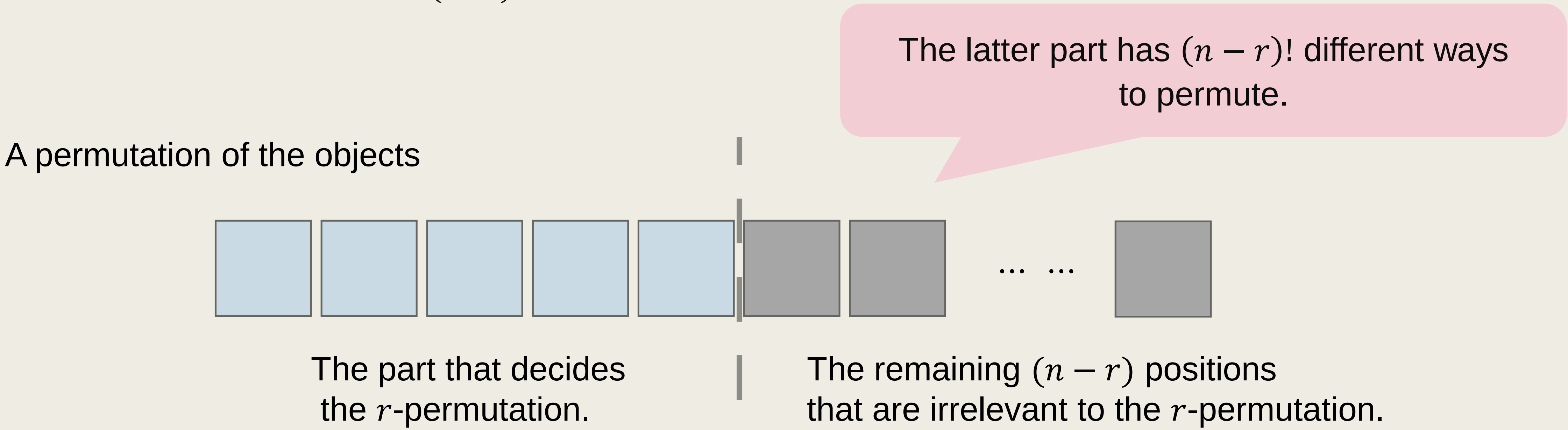
$$P(n,r) = \prod_{0 \le i < r} (n-i)$$

$$= \frac{n!}{(n-r)!}$$

- Alternatively, each r-permutation corresponds to exactly $(n-r)!$ different permutations of the objects.

  Out of the $n!$ permutations of the objects,
  every $(n-r)!$ of them uniquely decide an r-permutation.

  Hence, $P(n,r) = \frac{n!}{(n-r)!}$.



Combination

An r-combination of n distinct objects is an unordered selection, or, subset, of r objects out of the n objects.

- Let $C(n,r)$, or $\binom{n}{r}$, denote the number of different r-combinations of a set of n distinct objects.
  - Since an r-permutation can be obtained uniquely by first picking an r-combinations and then rearranging the r objects in any order.

$$P(n,r) = C(n,r) \times P(r,r), \quad \text{and} \quad C(n,r) = \binom{n}{r} = \frac{P(n,r)}{P(r,r)} = \frac{n!}{r!\,(n-r)!}.$$

Example 2. Poker Probabilities

a. How many 5-card hands (subsets) can be formed from a standard 52-card deck?

b. If a 5-card hand is chosen at random, what is the probability of obtaining a flush (all five cards in the hand are in the same suit) ?

c. What is the probability of obtaining three, but not four, Aces?

Example 2. Poker Probabilities

a. How many 5-card hands (subsets) can be formed from a standard 52-card deck?

$$C(52,5) = \binom{52}{5} = \frac{52!}{5!\,47!} = \frac{48 \times 49 \times 50 \times 51 \times 52}{5!} = 2{,}598{,}960\,.$$

Example 2. Poker Probabilities

b. If a 5-card hand is chosen at random, what is the probability of obtaining a flush (all five cards in the hand are in the same suit) ?

To form a flush, we first decide the suit and then pick 5 cards from that suit.

The total number of possible 5-card flushes is

$$\binom{4}{1} \times \binom{13}{5} = 4 \times \frac{13!}{5!\,8!} = 4 \times 1287 = 5148\,.$$

The probability is $5148 \,/\, 2{,}598{,}960 \approx 0.00198 \approx 0.2\%$.

Example 2. Poker Probabilities

c. What is the probability of obtaining three, but not four, Aces?

To count the number of hands with exactly 3 Aces,
we first pick three of the four Aces and then fill out the hand
with the remaining non-Ace cards.

This gives $\binom{4}{3} \times \binom{48}{2} = 4 \times 1128 = 4512.$

The probability is $\frac{4512}{2,598,960} \approx 0.00174.$

Example 3. Ranking Wizards

- How many ways are there to rank n candidates for the job of chief wizard?

  If the ranking is made at random (each ranking is equally likely to appear), what is the probability that the fifth candidate, Gandalf, is in second place?

Arrangement & Selections with Repetitions

Counting when some of the objects are identical

Theorem 1.

If there are n objects,

with r_1 of type 1, r_2 of type 2, ..., and r_m of type m,

where $r_1 + r_2 + \cdots + r_m = n$,

then the number of arrangements of these n objects,

denoted $P(n; r_1, r_2, \ldots, r_m)$, is

$$P(n; r_1, r_2, \ldots, r_m) = \binom{n}{r_1}\binom{n - r_1}{r_2}\binom{n - r_1 - r_2}{r_3} \cdots \binom{n - r_1 - r_2 - \cdots - r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1! \cdot r_2! \cdots r_m!} \, .$$

Proof - version 1.

- Form an arrangement as follows.
  - Pick from the n positions r_1 positions for objects of type 1.
  - Pick from the remaining $n - r_1$ positions r_2 positions for objects type 2.
  - Continue until the positions of all objects are decided.
- This gives

$$P(n; r_1, r_2, \dots, r_m) = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2-\cdots-r_{m-1}}{r_m}$$

$$= \frac{n!}{r_1! \cdot r_2! \cdots r_m!}\ .$$

By the same argument with $P(n, r)$

Proof - version 2.

- Treat the objects as distinct objects and consider the $n!$ permutations.
  - Each different positioning of objects type 1 contributes $r_1!$ permutations.
  - Each different positioning of objects type 2 contributes $r_2!$ permutations.
  - …. so on so forth.
- The number of arrangements is thus

$$P(n; r_1, r_2, \dots, r_m) = \frac{n!}{r_1! \cdot r_2! \cdots r_m!} .$$

Example 4. Arrangements of “banana”

- How many arrangements are there of the six letters b, a, n, a, n, a ?

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- How many arrangements are there of the six letters b, a, n, a, n, a ?

The six letters consist of 3 ‘a’,
1 ‘b’, and
2 ‘n’.

The number of arrangements is

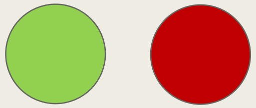
$$P(6;3,1,2) = \frac{6!}{3!\,1!\,2!} = 60.$$

<u>Theorem 2.</u>

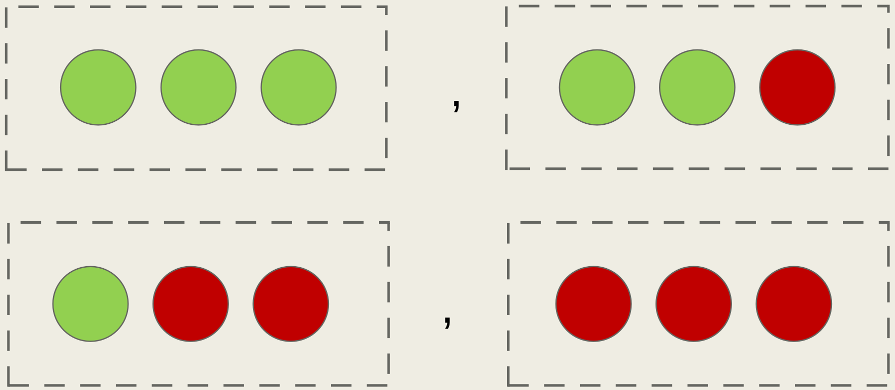
The number of selections with repetition of r objects chosen from n types of objects is $C(r+n-1,r)$.

For example,

to select with repetition 3 balls from two types of balls, green and red,



we have

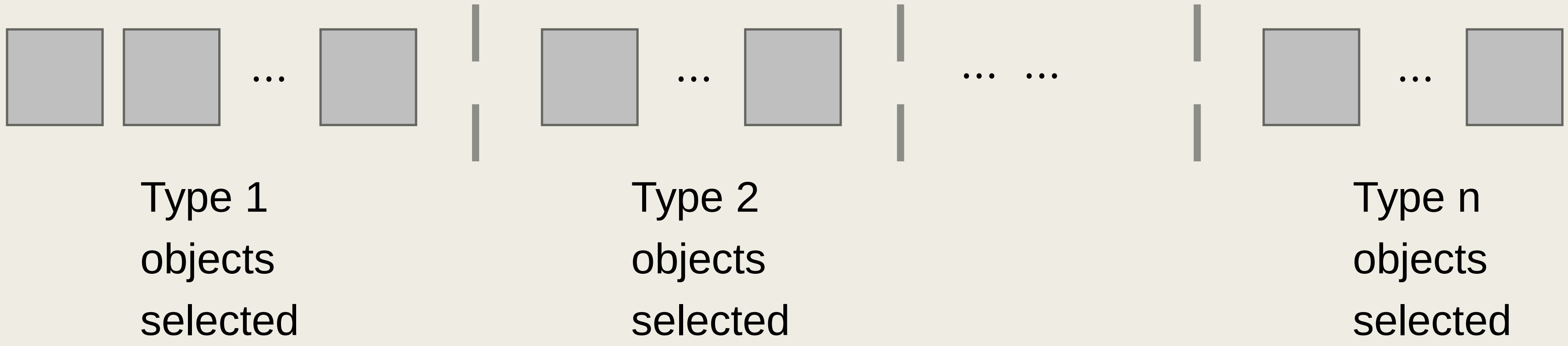


4 different ways in total!

Theorem 2.

The number of selections with repetition of r objects chosen from n types of objects is $C(r+n-1,r)$.

Represent each possible result as follows.



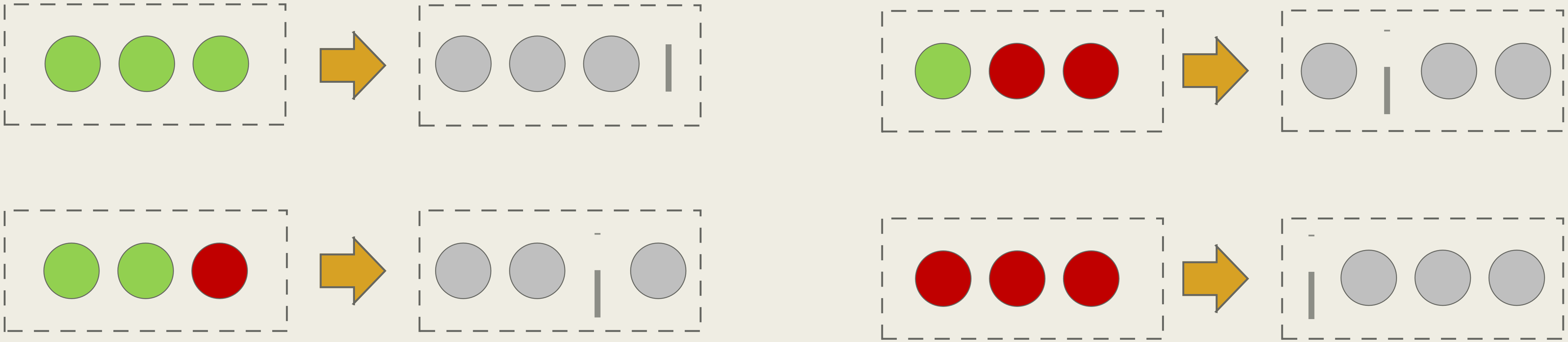
Hence, it is equivalent to arranging r identical and $n-1$ identical separators .

This gives $P(r+n-1;r,n-1)=C(r+n-1,r)$.

Theorem 2.

The number of selections with repetition of r objects chosen from n types of objects is $C(r+n-1, r)$.

For example, the result of selecting with repetition 3 balls from two types of balls can be represented as follows.



Example 5. Selections with Lower and Upper Bounds

- How many ways are there to pick 10 balls from a pile of red balls, blue balls, and purple balls, if

  a. There must be ***at least*** 5 red balls ?

  b. There must be ***no more than*** 5 red balls ?

Example 5. Selections with Lower and Upper Bounds

- How many ways are there to pick 10 balls from a pile of red balls, blue balls, and purple balls, if

  a. There must be ***at least*** 5 red balls ?

We first pick 5 red balls and then select with repetition 5 balls from the three types of balls.

It means that, we are in fact selecting 5 balls of 3 types with repetitions.

This gives $C(5+3-1,5) = \binom{7}{5} = 21$ ways.

Example 5. Selections with Lower and Upper Bounds

- How many ways are there to pick 10 balls from a pile of red balls, blue balls, and purple balls, if

b. There must be ***no more than*** 5 red balls ?

For the constraint of at most 5 red balls, we count the complementary set.

The answer is

$$\binom{10+3-1}{10} - \binom{4+3-1}{4} = 66 - 15 = 51 \text{ ways.}$$

Selection of 10 balls without restrictions

Selection of 10 balls with at least 6 red balls

Distributions

Distribution is equivalent to arrangement / selection with repetition.

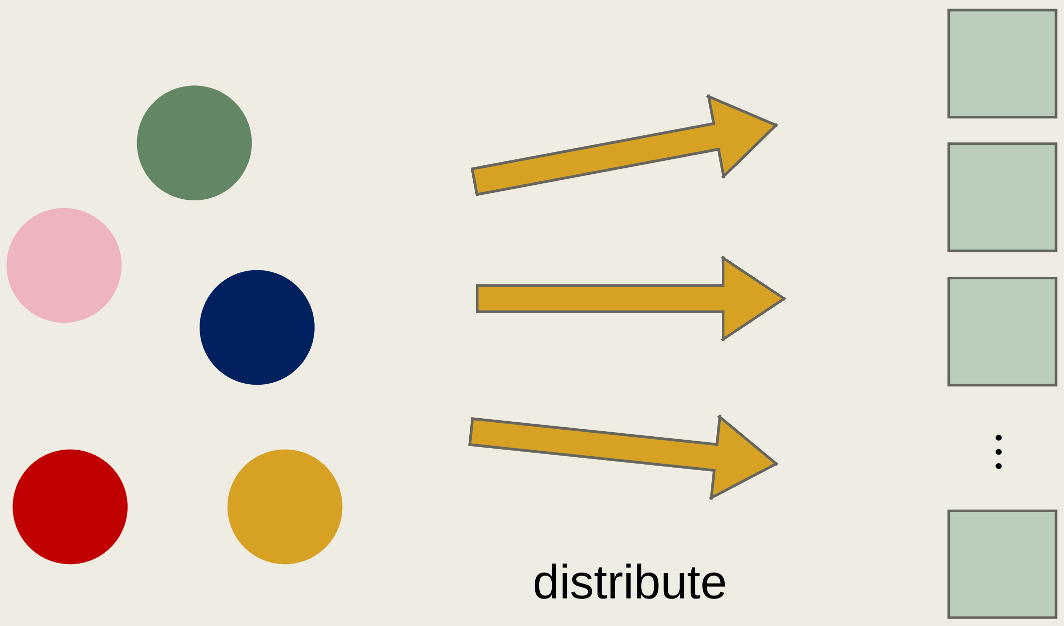
Distributions

Distributions of ***distinct objects*** are equivalent to *arrangements with repetitions*.

Distributions of ***identical objects*** are equivalent to *selections with repetitions*.

Distributions

Distributions of ***distinct objects*** are equivalent to *arrangements with repetitions*.



r distinct balls

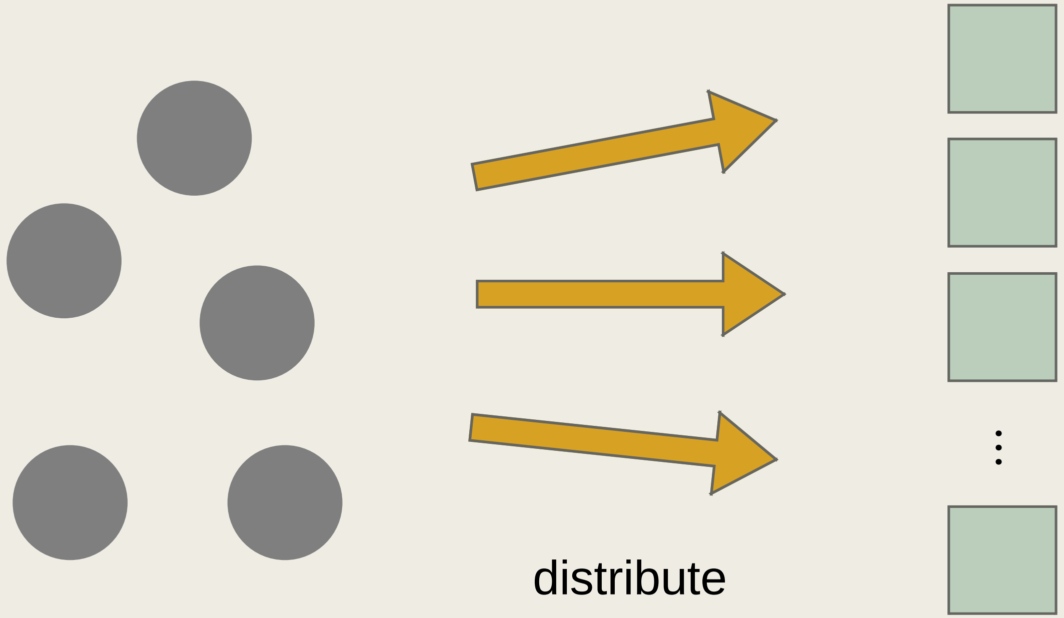
n distinct boxes

For each ball,
decide a destination.

In total, n^r different ways.

Distributions

Distributions of ***identical objects*** are equivalent to *selections with repetitions*.



r identical balls

n distinct boxes

Select r destinations of n types for the balls to go.

In total,
$C(r+n-1, r)$ different ways.

Equivalent Forms for Selection with Repetition

1. The number of ways to select r objects with repetition from n types of objects.
2. The number of ways to distribute r identical objects into n distinct boxes.
3. The number of nonnegative integer solutions to

$$x_1 + x_2 + \cdots + x_n = r\ .$$

Ways to Arrange, Select from, n Items or to Distribute r Objects into n Boxes

	Arrangement (Ordered Outcome) or Distributions of Distinct Objects	**Combinations (Unordered Outcome) or Distributions of Identical Objects**
No repetition	$P(n, r)$	$C(n, r)$
Unlimited repetition	n^r	$C(n + r - 1, r)$
Restricted repetition	$P(n; r_1, r_2, \dots, r_m)$	---

Binomial Identities

<u>The Double Counting Principle</u> for proving identities -

If the elements of a set are counted in two different ways, the answers are the same.

Binomial Theorem

$$(1+x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{k}x^k + \cdots + \binom{n}{n}x^n .$$

Consider the expansion of

$$(1+x)^n \;=\; (1+x)\cdot(1+x)\cdot\cdots\cdot(1+x) .$$

For any $0 \leq k \leq n$, we get an x^k, if x is chosen k times (or, 1 is chosen $n-k$ times).

Hence, the coefficient of x^k is

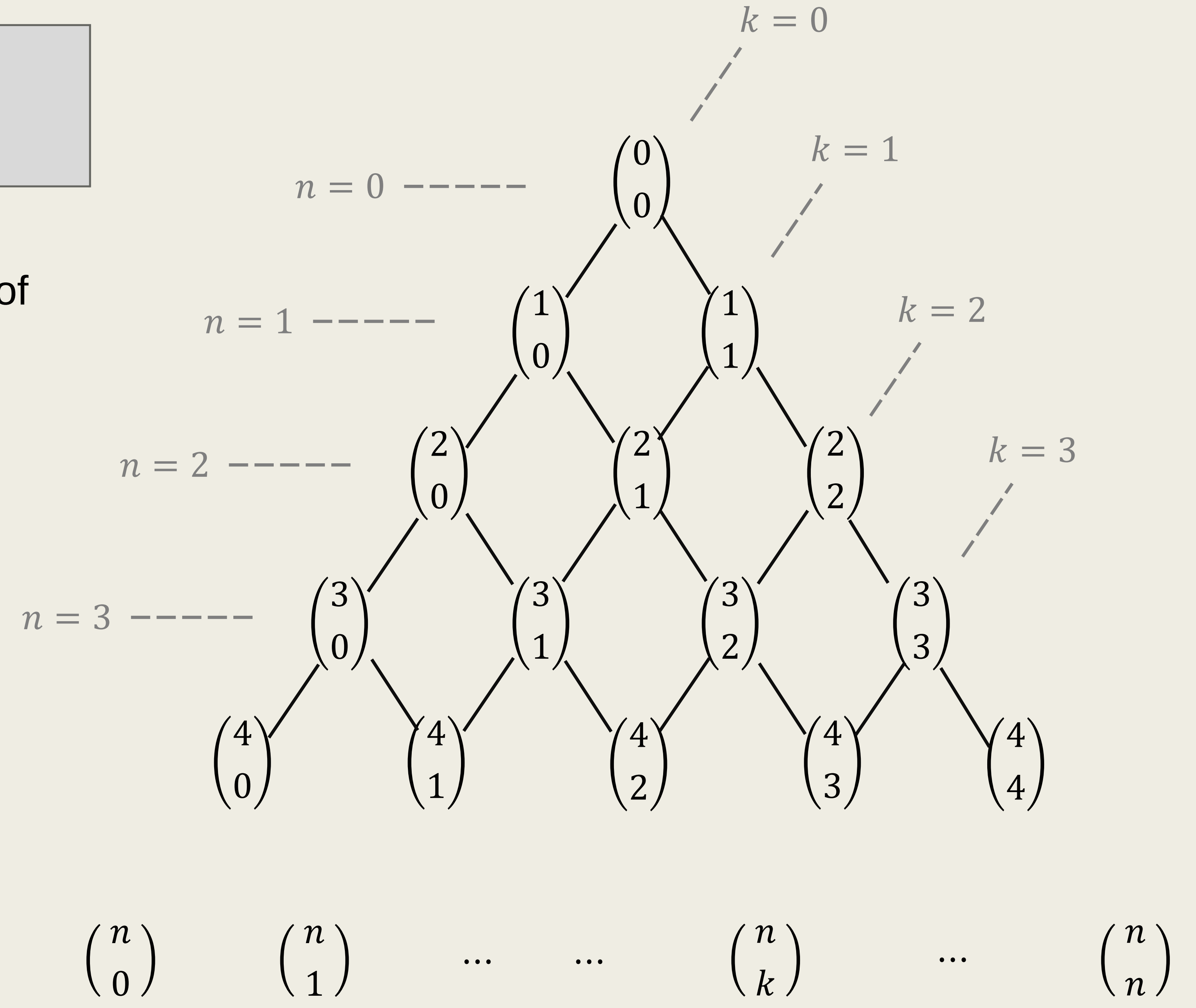
$$\binom{n}{k} = \binom{n}{n-k} .$$

The Pascal's Triangle

- Handy for proving a number of binomial identities.

- Denote the nodes by their coordinates (n, k).

- Any 'downward' path from $(0,0)$ to (n, k) must use $\begin{cases} k & \text{'R's} \\ n-k & \text{'L's} \end{cases}$.

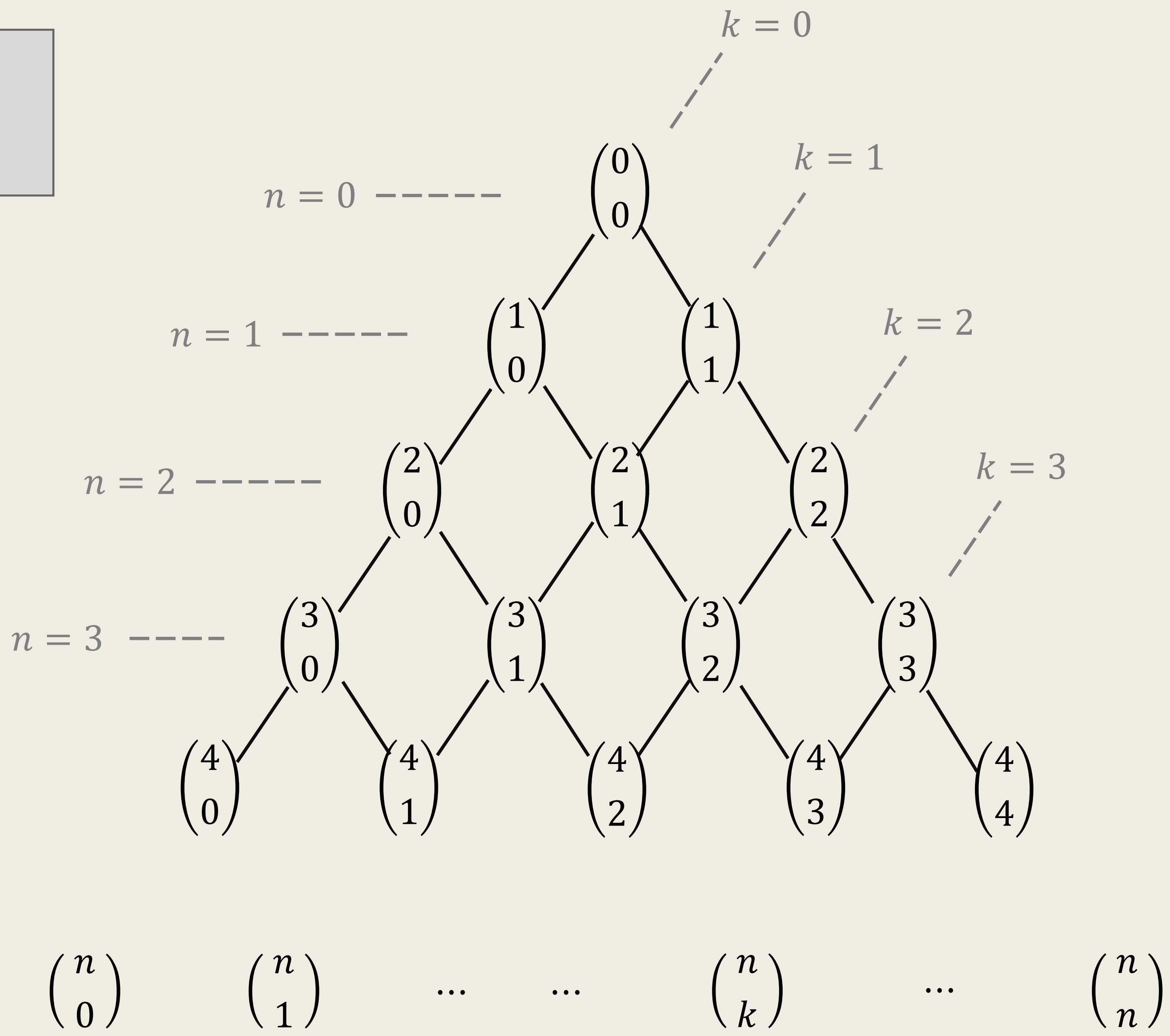


The Pascal's Triangle

- Any 'downward' path from $(0,0)$ to (n,k) must use $\begin{cases} k & \text{'R's} \\ n-k & \text{'L's} \end{cases}$.

- The number of downward paths from $(0,0)$ to (n,k) is $\binom{n}{k}$, i.e.,

  the number of arrangements with k 'R's and $n-k$ 'L's.



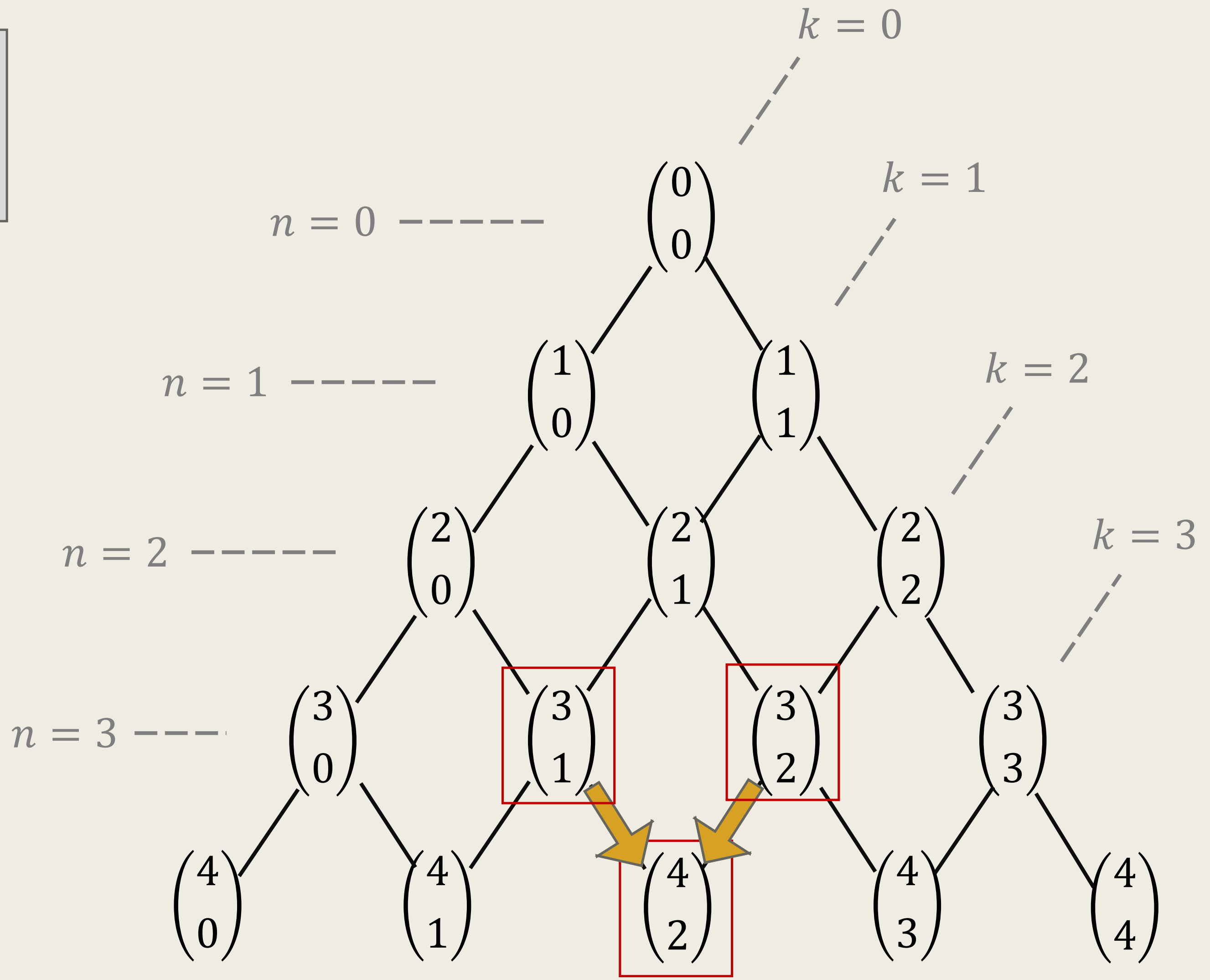
The Pascal's Triangle

- For any n, k,

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}.$$

- Any downward path to (n, k) must pass $(n-1, k)$ or $(n-1, k-1)$.

- The number of downward paths to (n, k) equals the number of paths to $(n-1, k)$ and $(n-1, k-1)$.



The downward paths to (n, k) is either a downward path to $(n-1, k-1)$ or $(n-1, k)$ before it enters (n, k).

The Pascal's Triangle

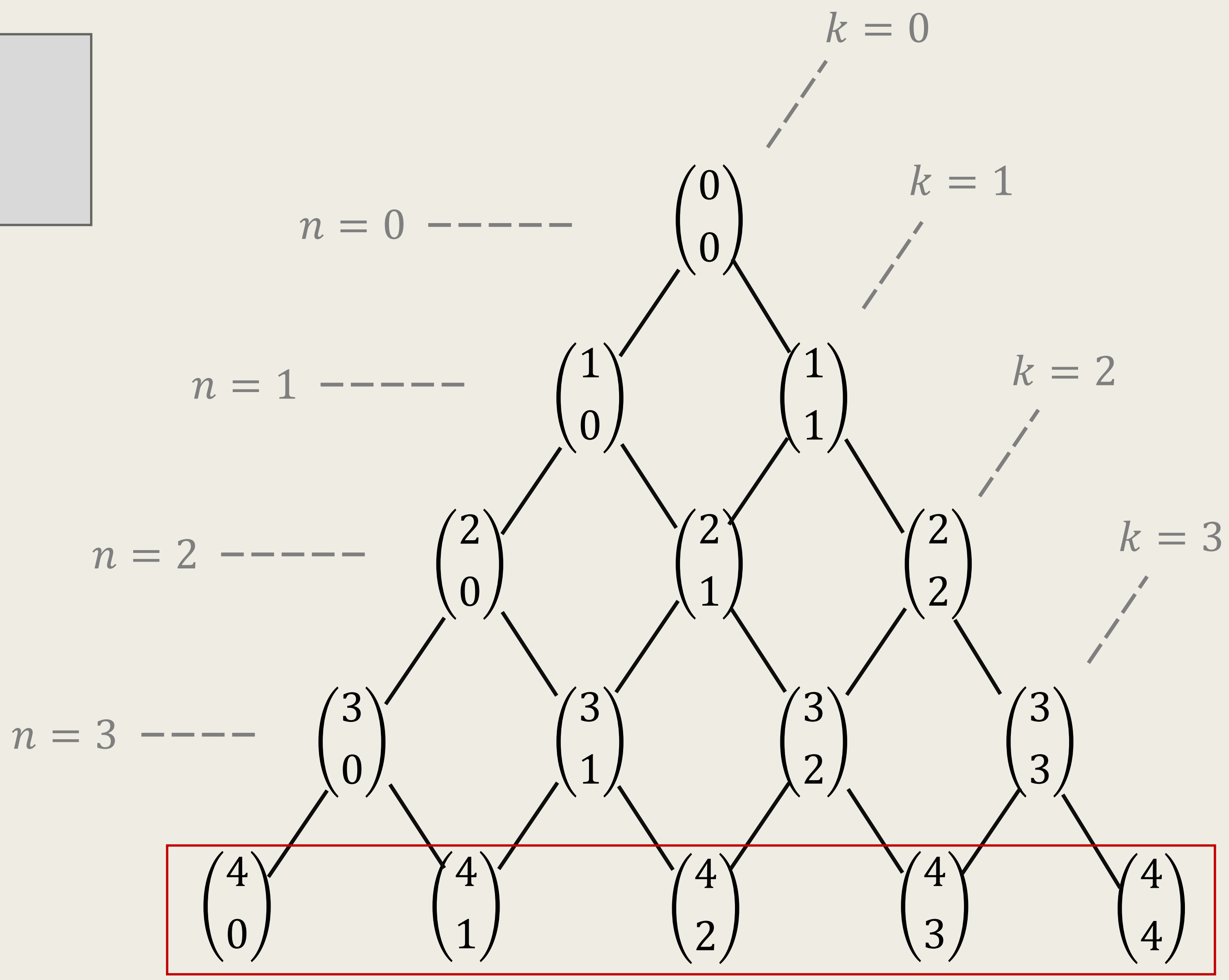
- For any $n \in \mathbb{Z}^{\geq 0}$,

$$\sum_{0 \leq k \leq n} \binom{n}{k} = 2^n .$$

- On the L.H.S.,
  the sum of the number of arrangements with k 'R's.

- On the R.H.S.,
  the total number of arrangements of 'L' and 'R'.

- ***By double-counting principle***,
  they are equal.



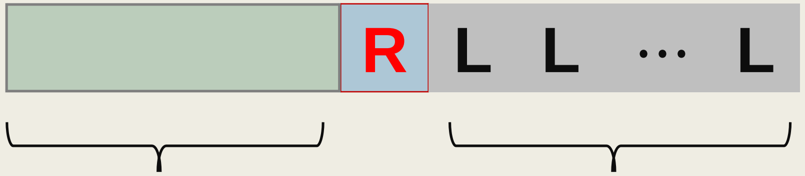
The identity counts all possible downward paths to the n^{th}-level and all length-n arrangements of 'L' and 'R'.

The Pascal's Triangle

- For any $n, r \in \mathbb{Z}^{\geq 0},\ n \geq r$,

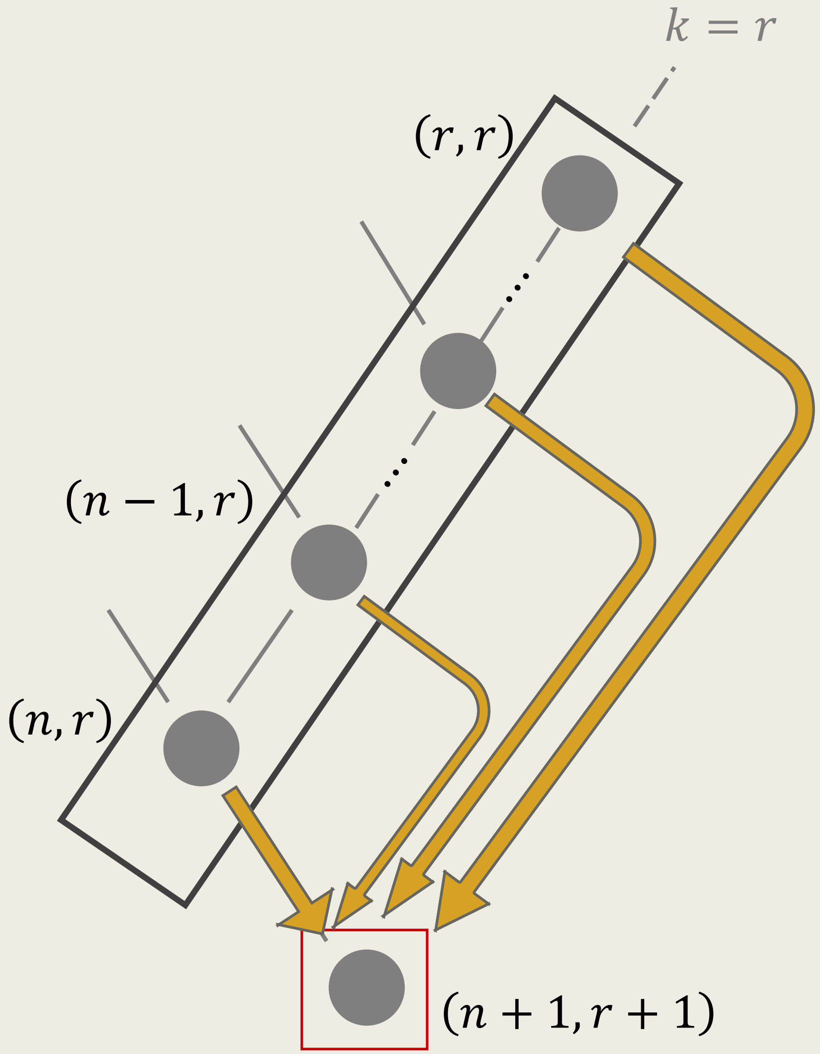
$$\sum_{0 \leq k \leq n-r} \binom{r+k}{r} = \binom{n+1}{r+1}.$$

- Distinguish any downward path to $(n+1, r+1)$ by **its last 'R'.**



A downward path to $(r+k, r)$ for some $0 \leq k \leq n-r$

Zero or more 'L's



The identity counts the number of downward paths to $(n+1, r+1)$ and holds ***by the double-counting principle***.

The Pascal's Triangle

- For any $n \in \mathbb{Z}^{\geq 0}$,

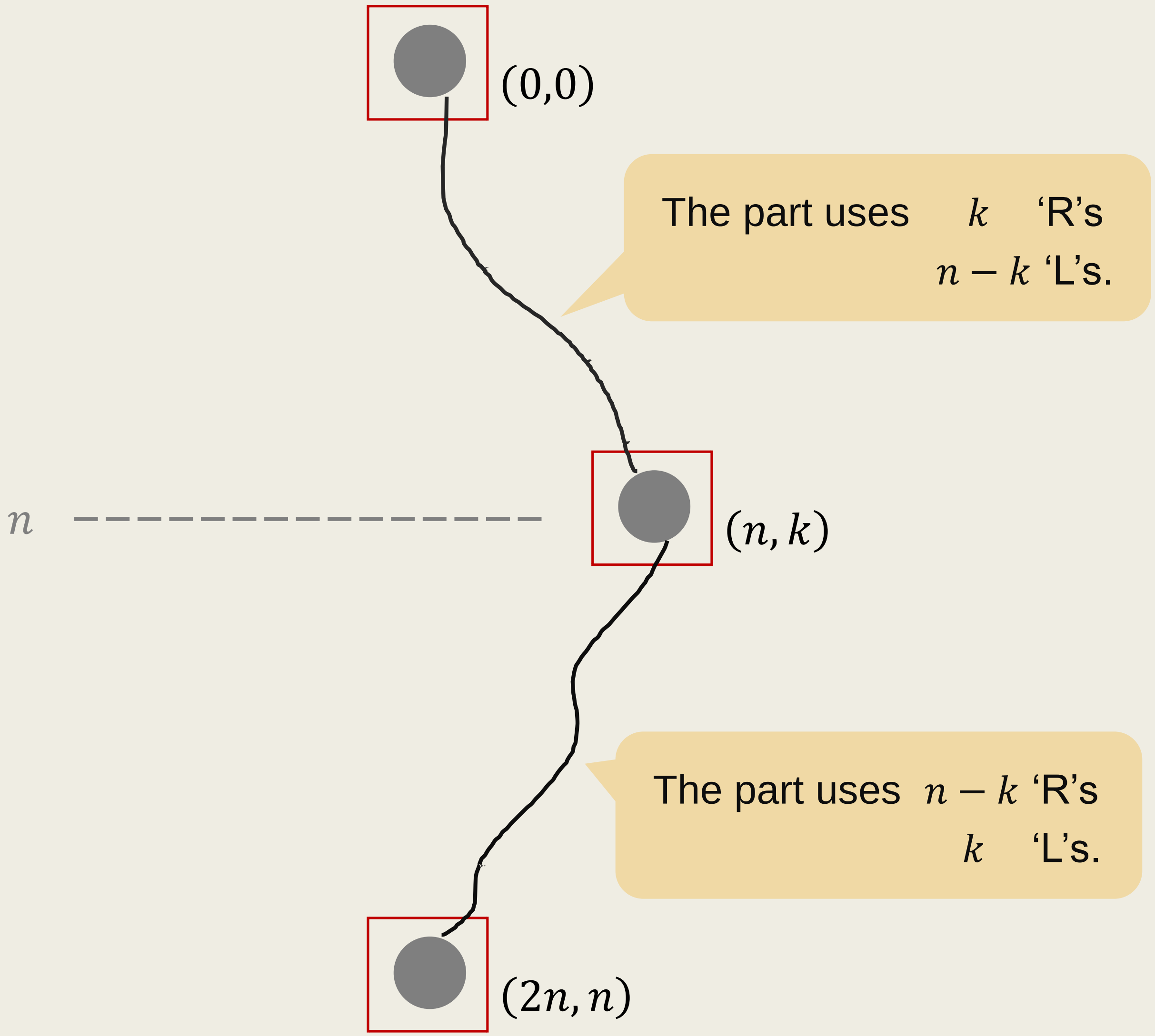
$$\sum_{0 \leq k \leq n} \binom{n}{k}^2 = \binom{2n}{n}.$$

- Consider a downward path to $(2n, n)$. Suppose that it passes the n^{th}-level via (n, k).

  There are

$$\binom{n}{k} \cdot \binom{n}{n-k} = \binom{n}{k}^2$$

  such downward paths.



The identity counts the number of downward paths to $(2n, n)$ and holds by the double-counting principle.

Principle of Inclusion-Exclusion

Counting with multiple non-exclusive constraints.

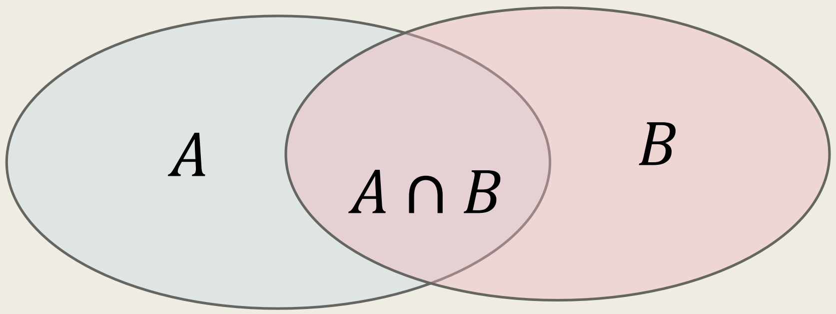
Counting with Venn Diagrams

- Let $N(A) \coloneqq |A|$ denote the number of elements in a set A.

- Suppose we have two sets A and B.

  We have

$$N(A \cup B) = N(A) + N(B) - N(A \cap B).$$



Why changed to intersection ?

Counting the intersection of the sets is usually easier than the union of them.

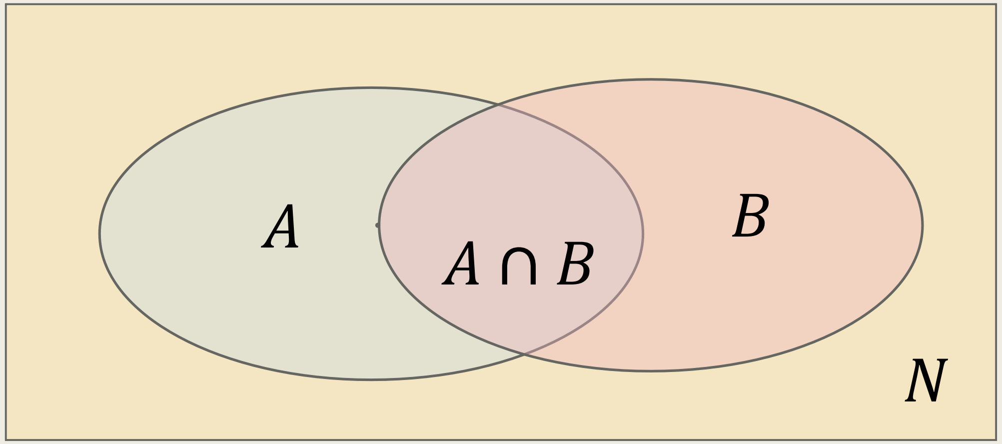
Counting with Venn Diagrams

- Suppose we have two sets A and B.

  We have $N(A \cup B) = N(A) + N(B) - N(A \cap B)$.

- Hence

$$N(\overline{A} \cap \overline{B}) = N(\overline{A \cup B}) = N - N(A \cup B)$$
$$= N - N(A) - N(B) + N(A \cap B).$$



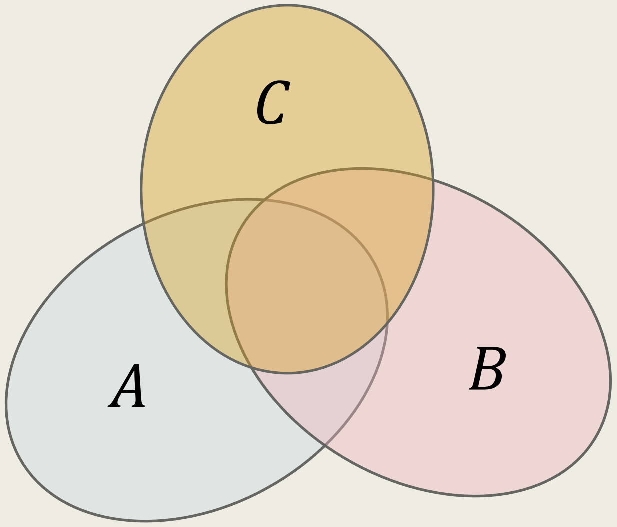
When A and B denote the undesirable events, $\overline{A} \cap \overline{B}$ corresponds to the desirable situation when none of the bad events has happened.

Counting with Venn Diagrams

- Suppose we have three sets A, B, and C.

  Then, counting the diagram, we have

$$N(A \cup B \cup C) = N(A) + N(B) + N(C)$$
$$-N(A \cap B) - N(B \cap C) - N(A \cap C)$$
$$+N(A \cap B \cap C)\,.$$



This gives the formula for $N\left(\overline{A} \cap \overline{B} \cap \overline{C}\right)$.

Example 6. Restricted Arrangements

- How many arrangements of the digits $0, 1, 2, \ldots, 9$ are there in which the first digit is greater than 1 and the last digit is less than 8 ?

Example 6. Restricted Arrangements

- How many arrangements of the digits $0, 1, 2, \ldots, 9$ are there in which the first digit is greater than 1 and the last digit is less than 8 ?

Let $A \coloneqq \{\text{arrangements with 0 or 1 in the first digit}\}$ and
$B \coloneqq \{\text{arrangements with 8 or 9 in the last digit}\}$.

Then, $|A| = \binom{2}{1} \times 9!$, $|B| = \binom{2}{1} \times 9!$, $|A \cap B| = \binom{2}{1} \times \binom{2}{1} \times 8!$, and

$$\left|\overline{A} \cap \overline{B}\right| = N - N(A) - N(B) + N(A \cap B)$$

$$= 10! - 2 \times (2 \times 9!) + (2 \times 2 \times 8!) = 58 \times 8!.$$

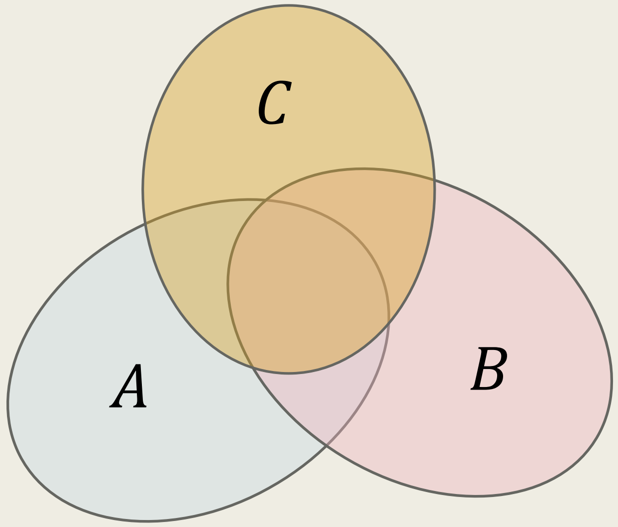
The definition of A and B makes the computation of $A \cap B$ easier.

The Inclusion-Exclusion Principle

- Let $A_1, A_2, \ldots, A_n \subseteq X$ be given sets.

  For any $I \subseteq \{1,2,\ldots,n\}$,

  define $A_I \coloneqq \bigcap_{i \in I} A_i$ with the convention that $A_\phi = X$.



- Let $A_1, A_2, \ldots, A_n \subseteq X$ be given sets. For any $I \subseteq \{1,2,\ldots,n\}$, define $A_I := \bigcap_{i \in I} A_i$ with the convention that $A_\phi = X$.

Theorem 3. (The inclusion-exclusion principle)

Let $A_1, A_2, \ldots, A_n$ be a sequence of sets. We have

$$\left| \bigcup_{1 \le i \le n} A_i \right| = \sum_{\substack{I \subseteq \{1,2,\ldots,n\}, \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot |A_I|$$

$$= \sum_{0 < k \le n} \sum_{\substack{I \subseteq \{1,2,\ldots,n\}, \\ |I| = k}} (-1)^{k+1} \cdot |A_I| .$$