

Combinatorial Mathematics

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Monday 18:30 – 20:20

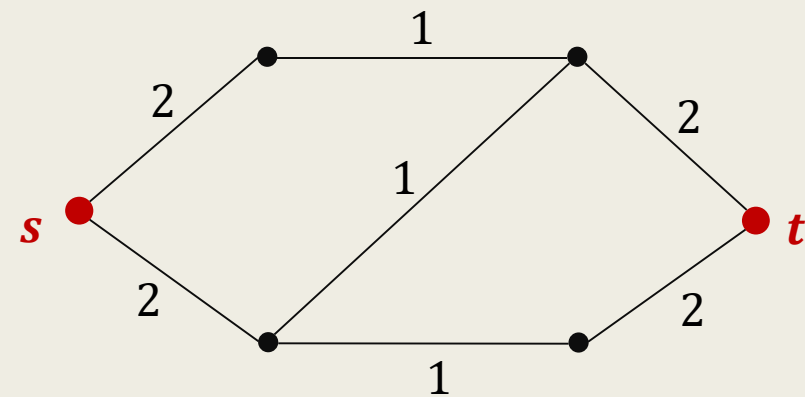
Outline

- The Network Flow & Minimum Cut
 - The Problem Model
 - The Weak-Duality between Max-Flow and Min-Cut
 - The Ford-Fulkerson Algorithm
- Some Efficient Algorithms for Max-Flow
 - Capacity Scaling, Edmonds-Karp
- Concluding Notes

The Network Flow Problem

Basic Definitions

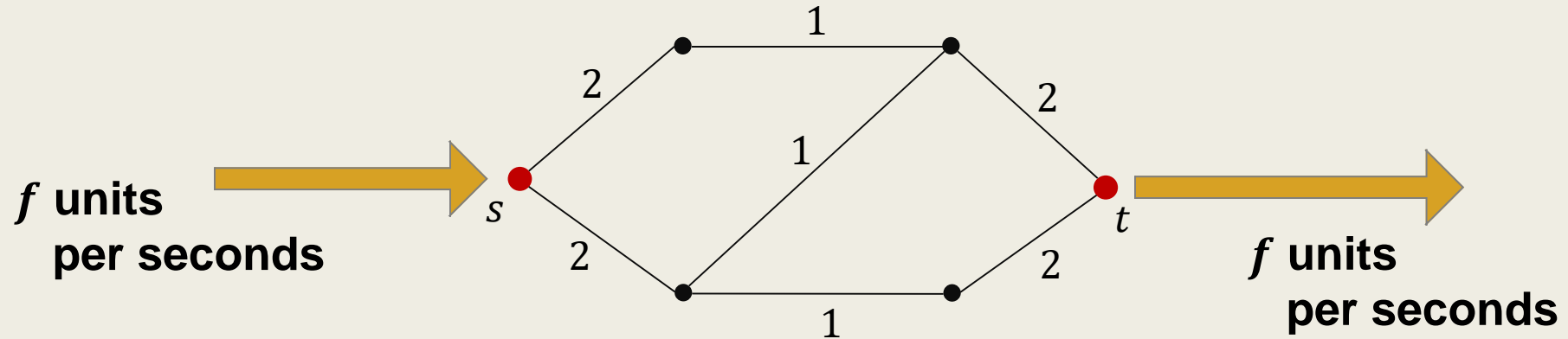
- A network is an undirected graph $G = (V, E)$ with
 - **Edge capacity** $c_{u,v} \in \mathbb{R}^{\geq 0}$ for each $(u, v) \in E$,
 - A **source vertex** $s \in V$, and
 - A **sink vertex** $t \in V$.



The network flow problem was originally defined on directed graphs. In this lecture, we assume undirected graphs for simplicity.

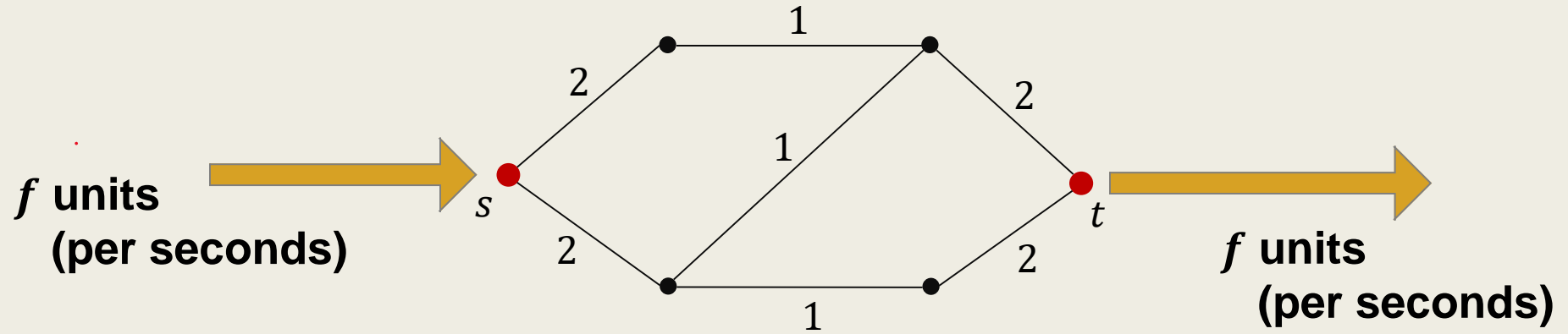
Flow can be *water*, *network packages*, *gasoline*, etc.

The Problem Model



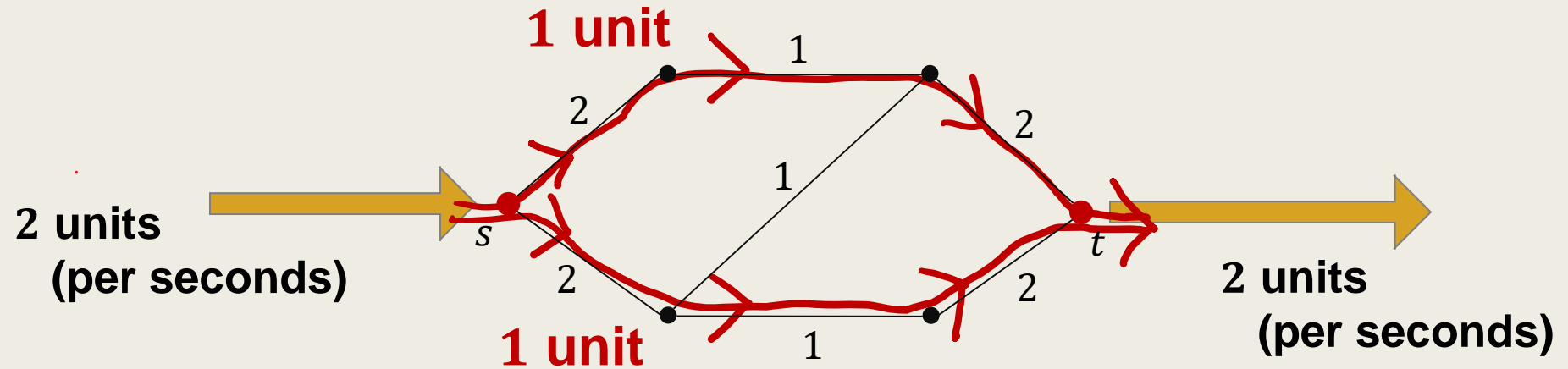
- **Flow** is *sent into* the network via *the source vertex* s , flowed through the pipes of the network, and then *exited* from the network via *the sink vertex* t .

The Problem Model



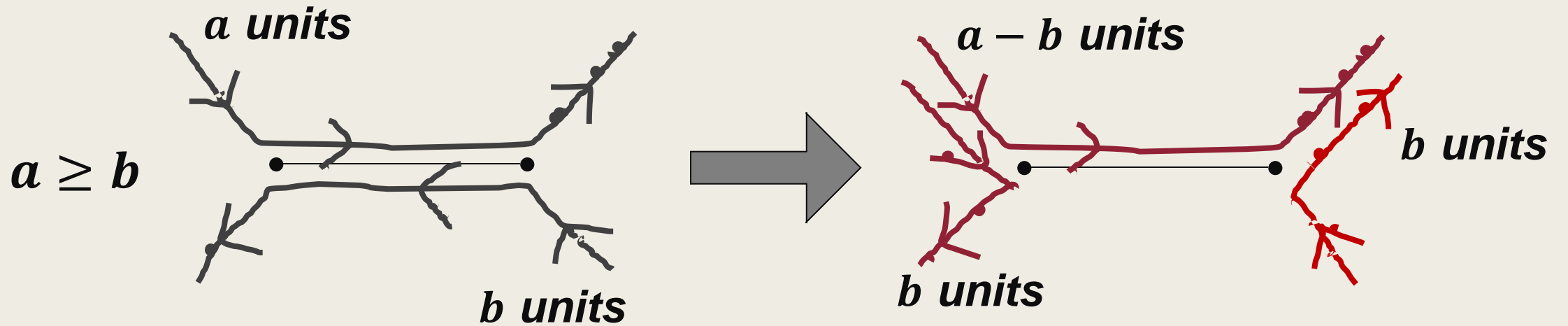
- The **edges** in the network are pipes *with limited capacity*, and allow flow to be sent *in either directions*.

The Problem Model



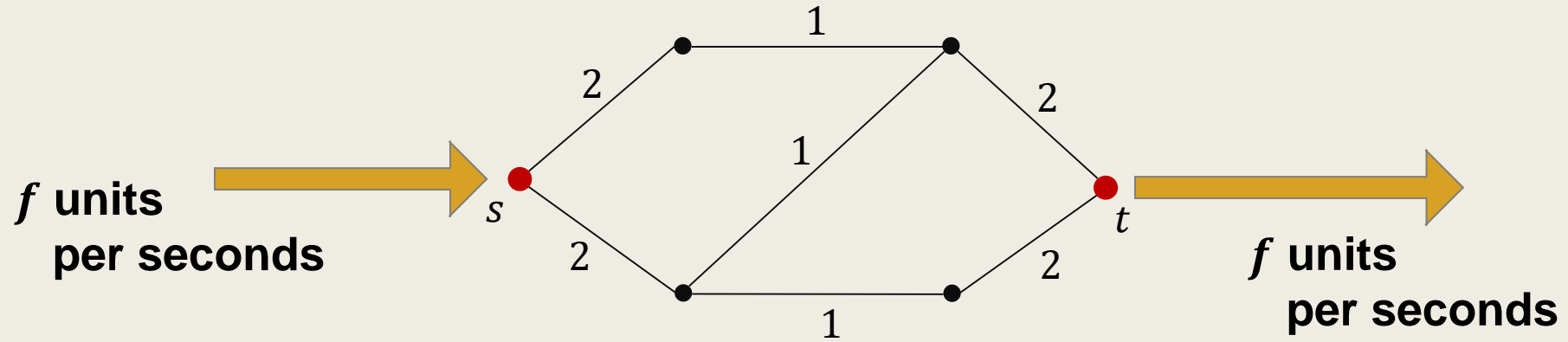
- We can decide how the flow goes in the network.

The Problem Model



- Flow sent from different directions on an edge cancels out.

The Problem Model



- **Question:**

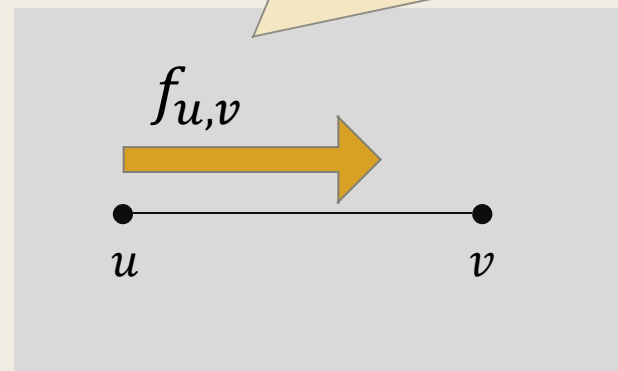
What is the ***maximum amount of flow*** that can be sent?

We use the function f to represent the flow in the network.

Formal Definition

- An s - t flow f is a function $f : V \times V \rightarrow \mathbb{R}$
such that...

$f_{u,v}$ is the amount of flow
that goes from u to v .



We use the function f to represent the flow in the network.

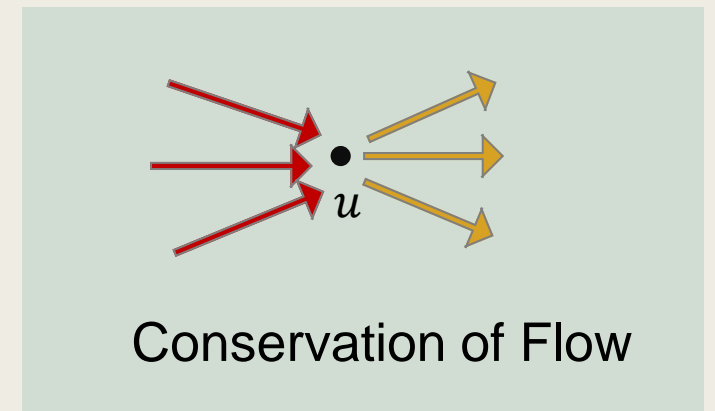
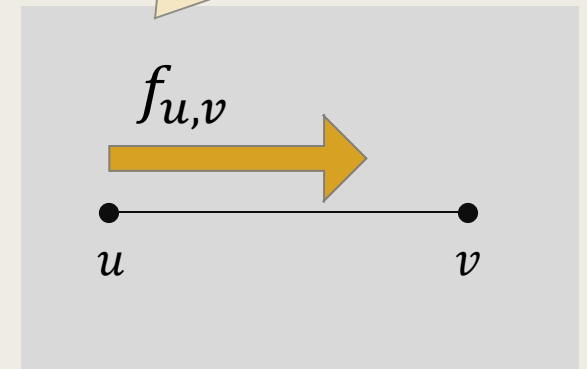
■ An s - t flow f is a function $f : V \times V \rightarrow \mathbb{R}$ such that

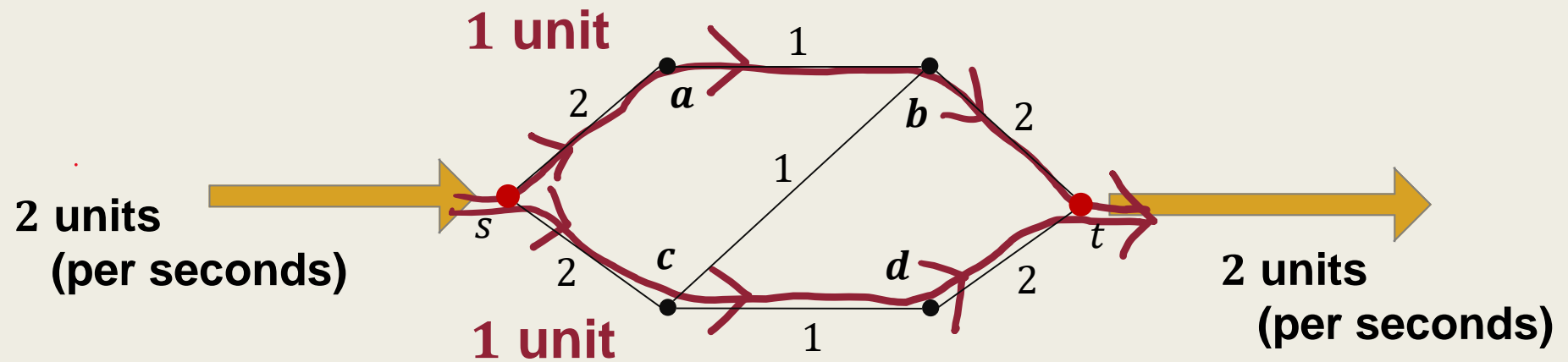
- $f_{u,v} = f_{v,u} = 0$, for all $(u, v) \notin E$.
- (Symmetry) $f_{u,v} = -f_{v,u}$, for all $(u, v) \in E$.
- (Capacity constraint) $|f_{u,v}| \leq c_{u,v}$, for all $(u, v) \in E$.
- (Conservation of flow) for any $u \in V \setminus \{s, t\}$,

$$\sum_{v:(u,v) \in E} f_{u,v} = 0.$$

- $f_{s,u} \geq 0$ and $f_{u,t} \geq 0$ for all $u \in V$.

The amount of flow that goes from u to v .





- In this example,

$$f_{s,a} = f_{a,b} = f_{b,t} = 1, \quad f_{s,c} = f_{c,d} = f_{d,t} = 1,$$

$$f_{a,s} = f_{b,a} = f_{t,b} = -1, \quad f_{c,s} = f_{d,c} = f_{t,d} = -1.$$

Formal Definition

- The value of a flow function f is defined as

$$\text{val}(f) := \sum_{v:(s,v) \in E} f_{s,v} \cdot$$

- By the conservation constraint, $\text{val}(f)$ is also equal to

$$\sum_{v:(v,t) \in E} f_{v,t} \cdot$$

The Maximum s - t Flow Problem

- Input :

- A graph / flow network $G = (V, E)$ with edge capacity $c_{u,v} \in \mathbb{R}^{\geq 0}$ for all $(u, v) \in E$ and a source-sink pair $s, t \in V$.

- Output :

- A flow function $f : E \rightarrow \mathbb{R}^{\geq 0}$ that has the maximum value among all possible s - t flows for G .
 - That is, $\text{val}(f) \geq \text{val}(f')$ holds for all s - t flow f' for G .

Notations

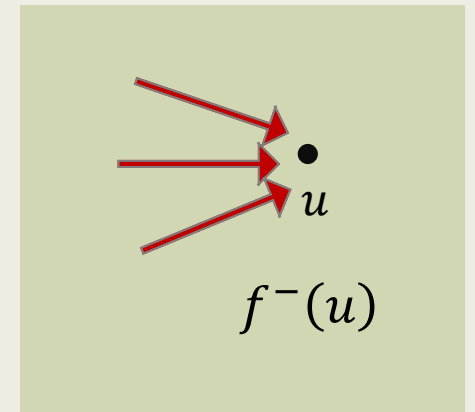
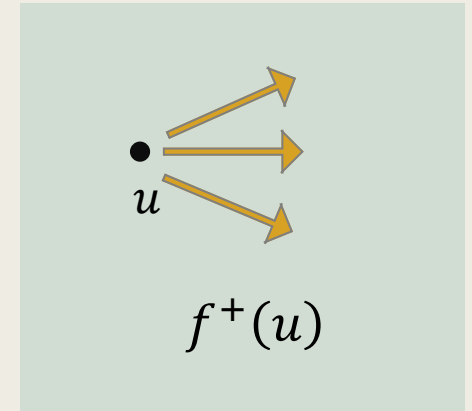
- For any vertex $u \in V$, we use

$$f^+(u) := \sum_{\substack{v:(u,v) \in E \\ f_{u,v} > 0}} f_{u,v}$$

to denote the total amount of flow leaving the vertex u .

- Similarly, we use $f^-(u) := \sum_{\substack{v:(u,v) \in E \\ f_{v,u} > 0}} f_{v,u}$ to denote

the total amount of flow entering the vertex u .



Notations

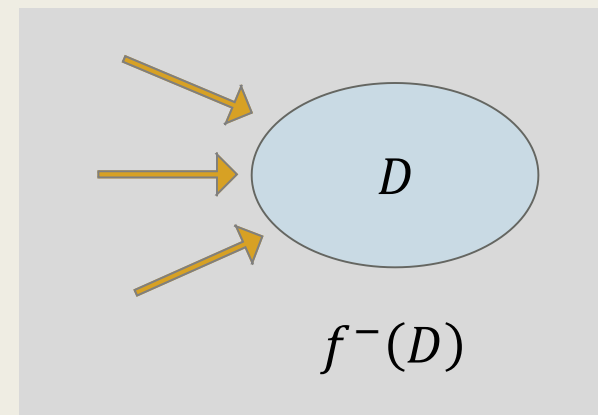
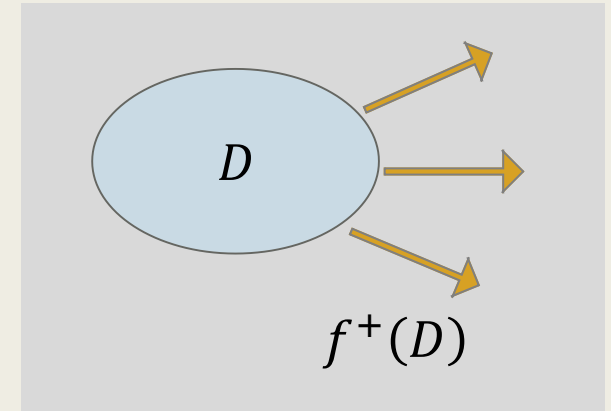
- For any $D \subseteq V$, we use

$$f^+(D) := \sum_{\substack{u \in D, v \in V \setminus D \\ f_{u,v} > 0}} f_{u,v}$$

to denote the total amount of flow leaving the vertex set D .

- Similarly, we use $f^-(D) := \sum_{\substack{u \in D, v \in V \setminus D \\ f_{v,u} > 0}} f_{v,u}$ to denote

the total amount of flow entering the vertex set D .

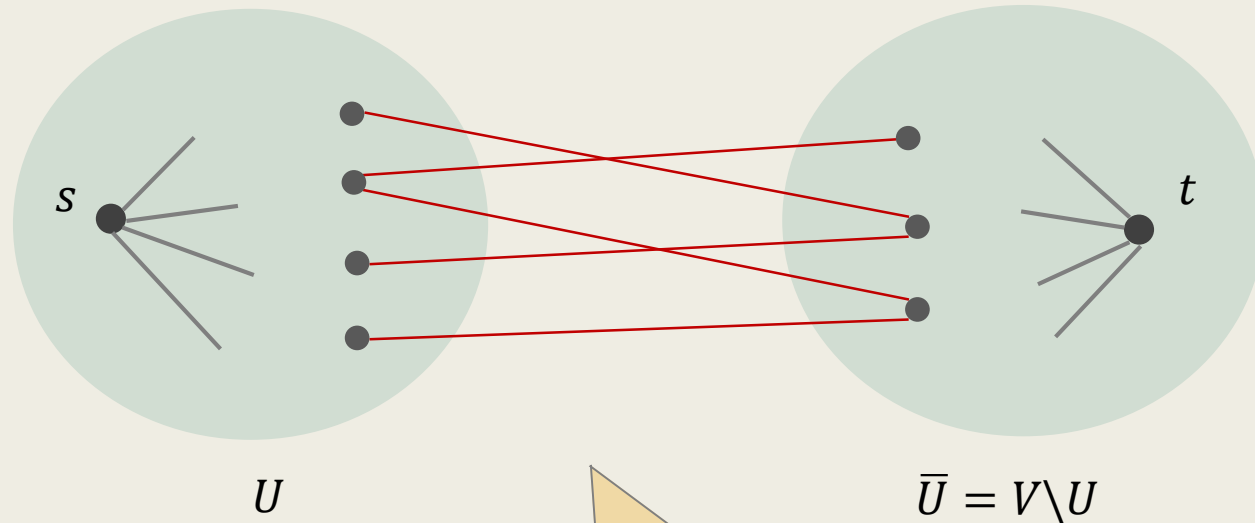


The Minimum s - t Cut Problem

The Cut for a Flow Network

- Let $G = (V, E)$ be a flow network with edge capacity (weight) $c_{u,v} \in \mathbb{R}^{\geq 0}$ for all $(u, v) \in E$ and a source-sink pair $s, t \in V$.
- **Definition.** (*s-t* Cut)
 - An *s-t* cut $C = [U, \bar{U}]$ is a partition of V into two sets U, \bar{U} such that $s \in U$ and $t \in \bar{U}$.
 - Conventionally, the *s-t* cut $[U, \bar{U}]$ can also be referred to as the edges between U and \bar{U} , depending on the context.

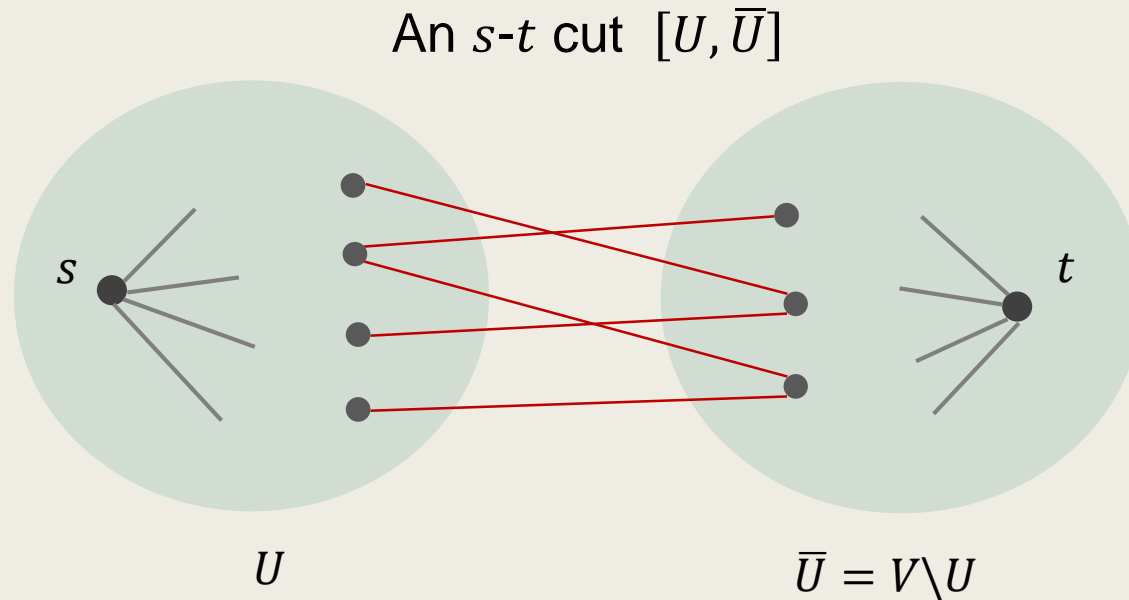
An s - t cut $[U, \bar{U}]$



Intuitively,
an s - t cut is a set of edges,
whose removal disconnects s from t .

- The **weight of a cut** $C = [U, \bar{U}]$ is defined to be the total weight (capacity) of the edges between U and \bar{U} .
 - That is,

$$w(C) = \sum_{e \in C} c_e .$$



The Minimum s - t Cut Problem

- Input :

- A graph $G = (V, E)$ with capacity (weight) $c_{u,v} \in \mathbb{R}^{\geq 0}$ for all $(u, v) \in E$ and a source-sink pair $s, t \in V$.

- Output :

- An s - t cut C for G that has the minimum weight among all possible s - t cut for G .
 - That is, $w(C) \leq w(C')$ holds all s - t cut C' for G .

The Weak Duality between Maximum Flow & Minimum Cut

The maximum $s-t$ flow is always bounded by the minimum $s-t$ cut.

Lemma 1. (Weak Duality between Flows and Cuts)

Let $G = (V, E)$ be a graph with edge capacity $c_e \in \mathbb{R}^{\geq 0}$ for all $e \in E$ and a source-sink pair $s, t \in V$,

f be an s - t flow and $C = [U, \bar{U}]$ be an s - t cut for G .

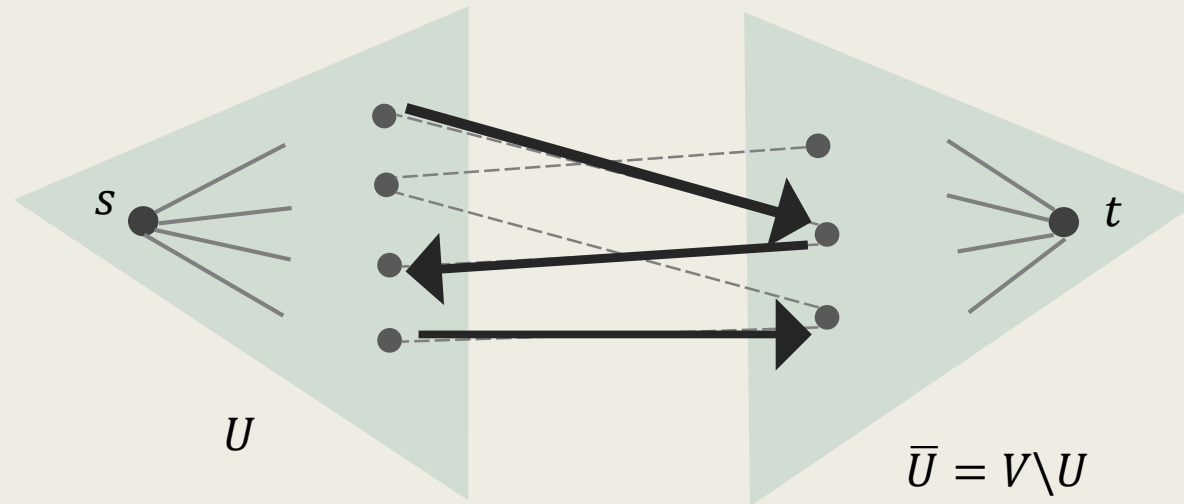
Then, $\text{val}(f) \leq w(C)$, i.e.,

$$\sum_{v \in V: (s,v) \in E} f_{s,v} \leq \sum_{e \in C} c_e .$$

- The proof for Lemma 1 is straightforward.

- We have

$$\text{val}(f) = f^+(U) - f^-(U) \leq f^+(U) \leq \sum_{e \in [U, \bar{U}]} c_e.$$



Remarks.

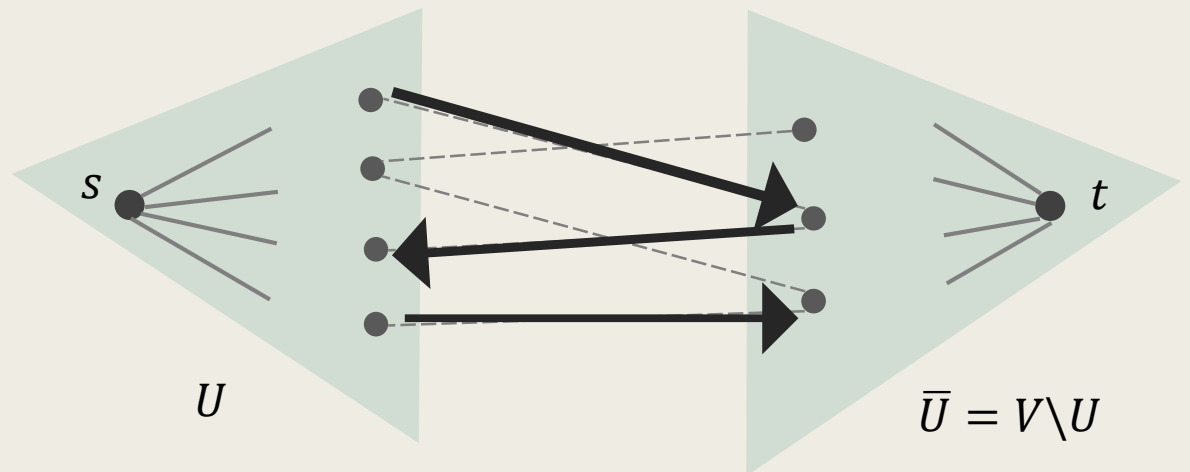
- Lemma 1 implies that,
 - If $\text{val}(f) = w(C)$ holds for some f and C , then they are both optimal.
 - In this case, we say that f and C *witnesses* the optimality of each other.

The Residual Network G_f and

The Ford-Fulkerson Algorithm

Computing the Maximum Flow

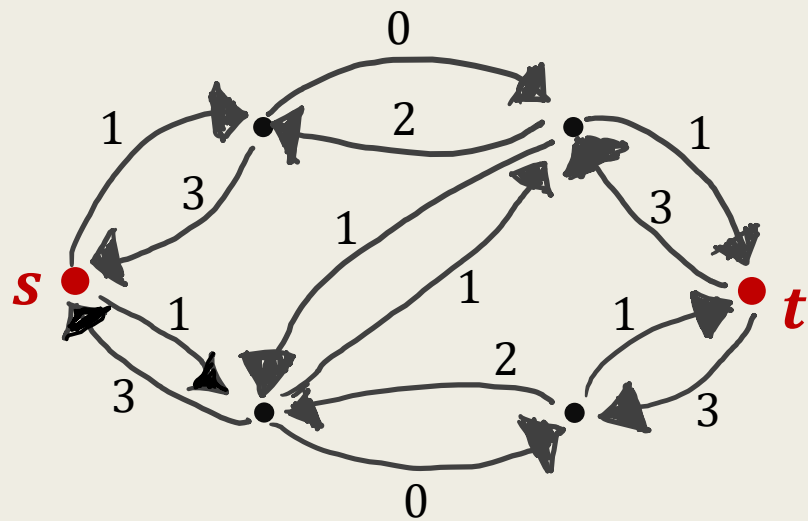
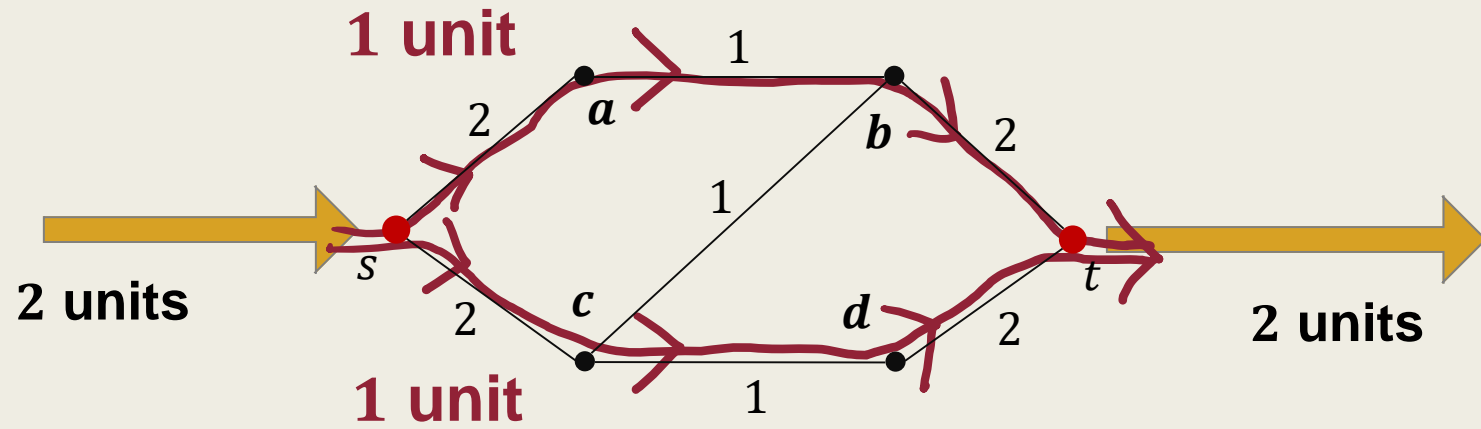
- A simple greedy algorithm
 - Start with a trivial flow $f = 0$.
 - Iteratively augment the current flow to make the value larger, until no more flow can be sent.
 - Then, we have a cut with an equal weight for the network.



The Residual Graph G_f

- Let f be a flow function for the input graph G .
 - Define the residual graph $G_f = (V, E_f)$ to be the directed graph with
 - Vertex set V ,
 - (Directed) Edge set $E_f := \{ (u, v) : (u, v) \in E \}$,
 - Capacity $c_f(u, v) := c_{u,v} - f_{u,v}$, for each $(u, v) \in E_f$.

Intuitively, $c_f(u, v)$ is the remaining capacity on the directed edge (u, v) .



Augmenting Paths in the Residual Graph G_f

- Let G_f be a residual graph with edge capacity c_f .
 - An s - t path

$$P = v_0 v_1 v_2 \cdots v_k$$

with $s = v_0$ and $t = v_k$ is said to be an f -augmenting path if

- $c_f(v_i, v_{i+1}) > 0$, for all $0 \leq i < k$.

The ***residual capacity*** along the path is > 0 .

- Let G_f be a residual graph with edge capacity c_f .

- An s - t path

$$P = v_0 v_1 v_2 \cdots v_k$$

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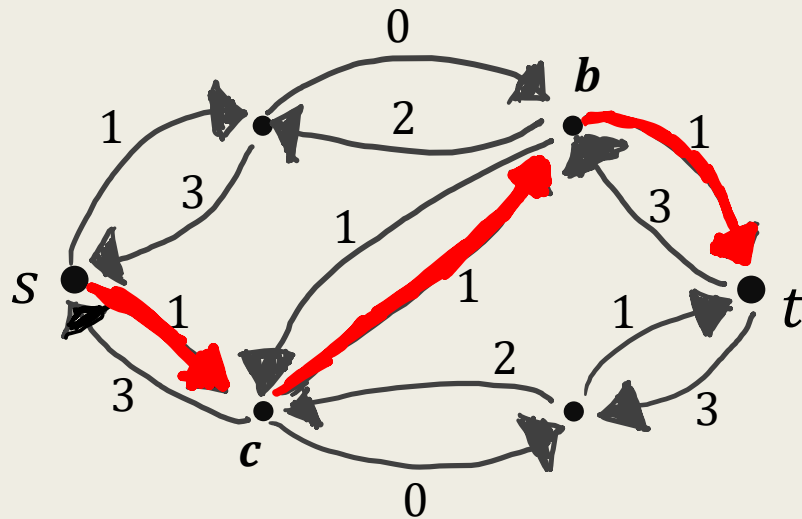
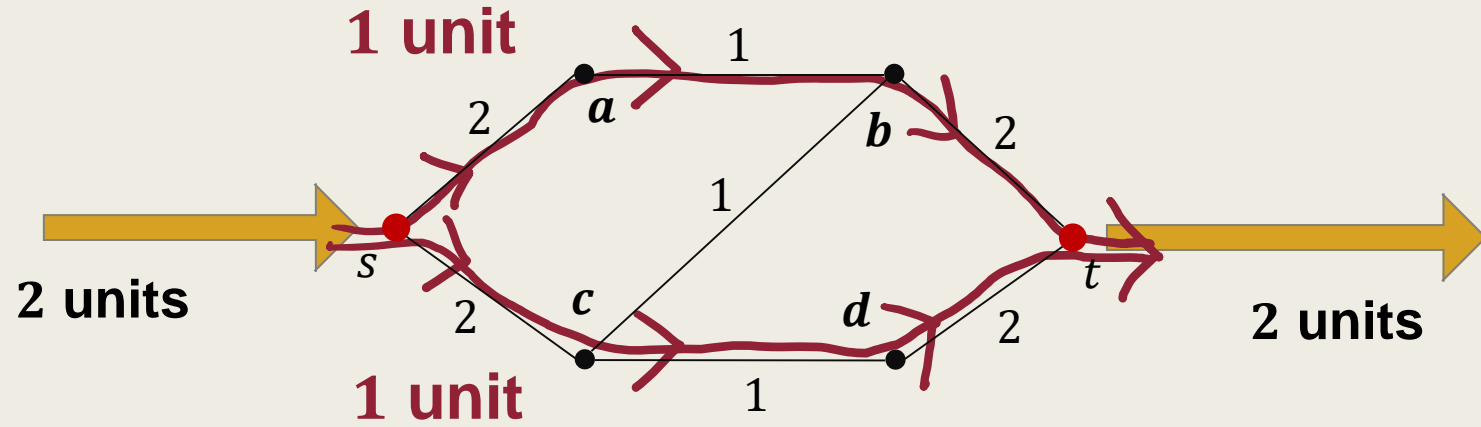
The **residual capacity** along the path is > 0 .

- Define

$$\Delta_P := \min_{0 \leq i < k} c_f(v_i, v_{i+1})$$

to be the minimum capacity along the path P in G_f .

An extra flow with value Δ_P can be sent along P .



$P = scbt$ with $\Delta_P = 1$

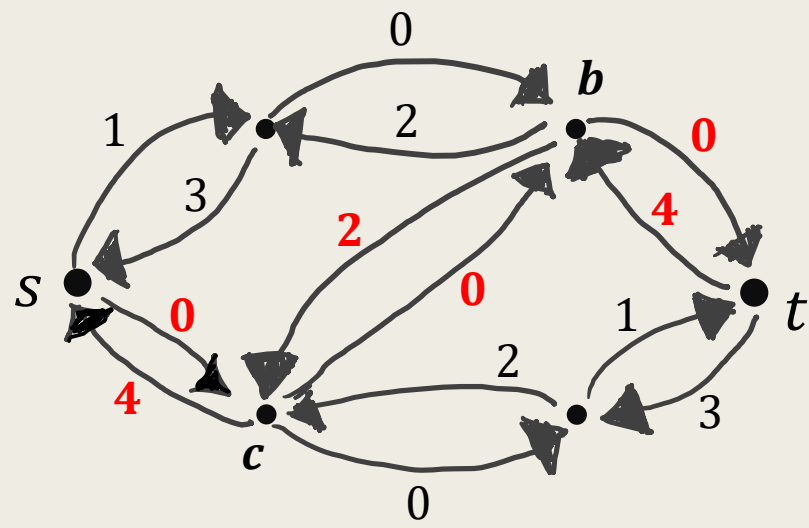
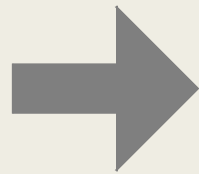
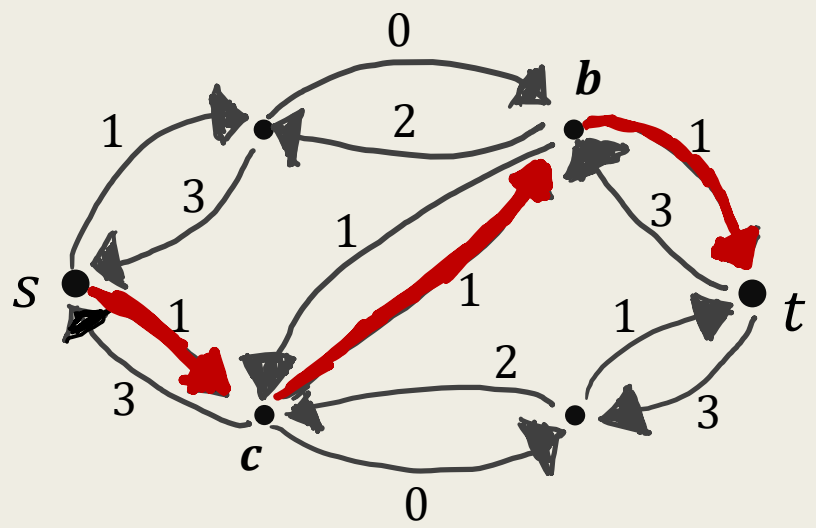
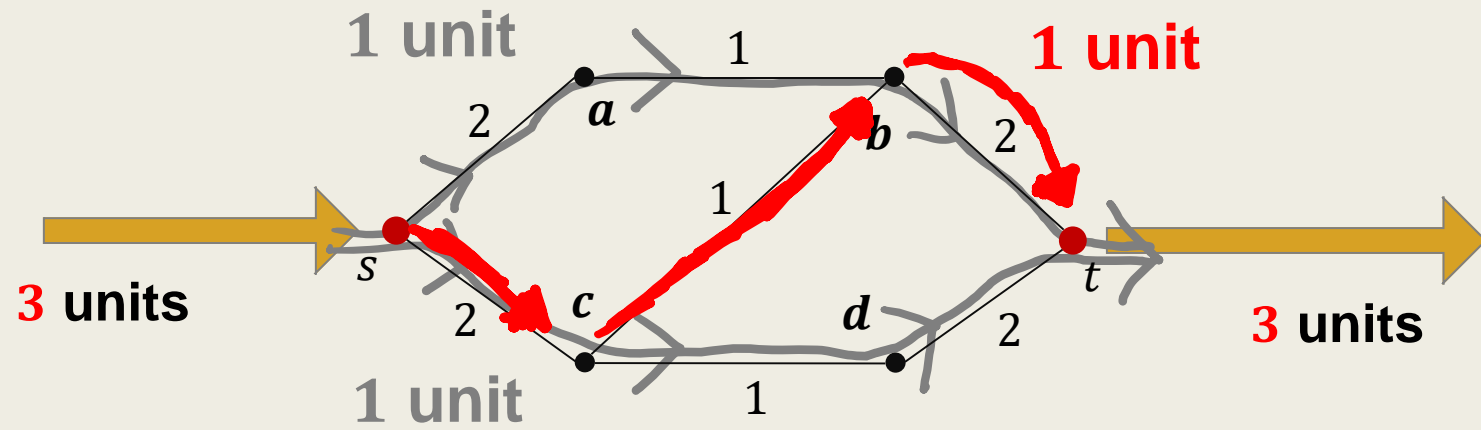
The value of f can be increased by $\Delta_P = 1$, by sending one unit of flow along $P = scbt$.

Updating the Residual Graph G_f

- Let G_f be a residual graph with edge capacity c_f and $P = v_0 v_1 v_2 \cdots v_k$ be an f -augmenting path with value Δ_P .
- Then, the value of f can be increased by Δ_P , and G_f is updated as follows.
 - For all $0 \leq i < k$,
decrease $c_f(v_i, v_{i+1})$ by Δ_P and increase $c_f(v_{i+1}, v_i)$ by Δ_P .

The residual capacity
in this direction is used.

More flow can be sent from the opposite direction.
So, the residual capacity in this direction increases.



The Ford-Fulkerson Algorithm for Max-Flow

- Start with a trivial flow $f = 0$.
- Repeat the following until there exists no f -augmenting path in G_f .
 - Compute an f -augmenting path P in G_f .
 - Use P to increase f by Δ_P .
- Output f .

A Slightly More-Detailed Pseudo-Code

- $f \leftarrow 0$.
resCap(u, v) = resCap(v, u) = $c_{u,v}$ for all $(u, v) \in E$.
- While there exists an f -augmenting path P in G_f , do
 - For each edge $(a, b) \in P$,
 - Decrease resCap(a, b) by Δ_P ,
 - Increase resCap(b, a) by Δ_P .
 - Increase f by Δ_P .
- Output f .

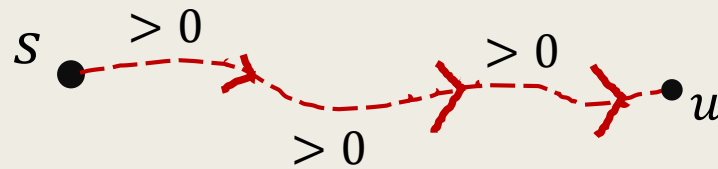
The Correctness of the Algorithm

- To prove that the Ford-Fulkerson algorithm computes a maximum s - t flow for the input graph G ,
 - We show that, when there exists no f -augmenting path in G_f , G has an s - t cut C with weight $\text{val}(f)$.
 - By Lemma 1, both C and f are optimal.

The Correctness of the Algorithm

- Suppose that there exists no f -augmenting path in G_f .
- Let U be the set of vertices that are reachable from s via paths with positive residual capacity in G_f .
 - That is,

$$U = \{ u \in V : \exists s - u \text{ path } P \text{ such that } c_f(a, b) > 0 \text{ for all } (a, b) \in P \}.$$

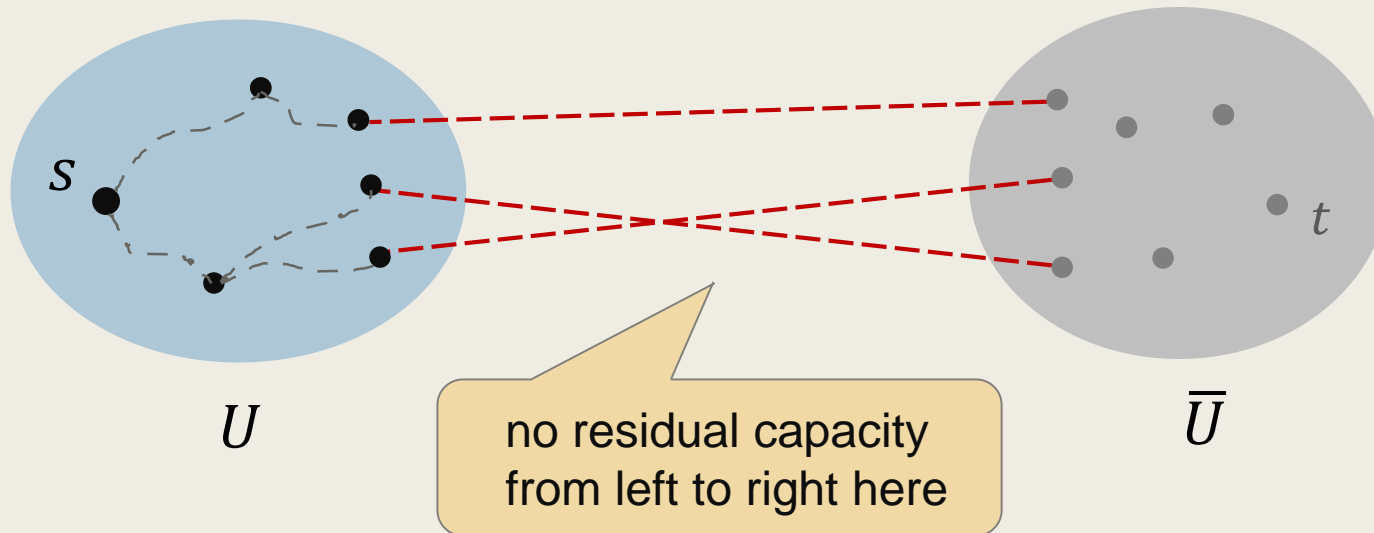


U does not contain the sink t .

$[U, \bar{U}]$ is an s - t cut.

- Suppose that there exists no f -augmenting path in G_f .
- Let U be the set of vertices that are reachable from s via paths with positive residual capacity in G_f .
 - Let $\bar{U} = V \setminus U$.
 - Then, $c_f(a, b) = 0$, for all $(a, b) \in [U, \bar{U}]$.

$f(a, b) = c_{a,b}$ for all $(a, b) \in [U, \bar{U}]$.



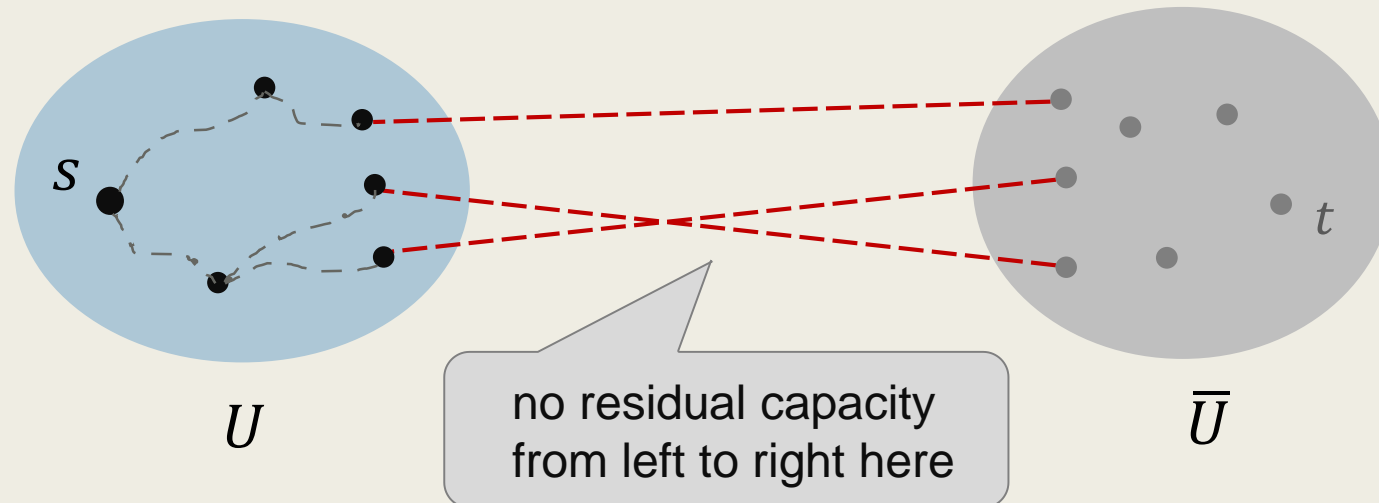
- Let $\bar{U} = V \setminus U$.

$$f(a, b) = c_{a,b} \text{ for all } (a, b) \in [U, \bar{U}].$$

- Then, $c_f(a, b) = 0$, for all $(a, b) \in [U, \bar{U}]$.

- Hence,

$$\text{val}(f) = f^+(U) = \sum_{\substack{a \in U \\ b \in \bar{U}}} f(a, b) = \sum_{\substack{a \in U \\ b \in \bar{U}}} c_{a,b} = w([U, \bar{U}]).$$



Time Complexity of the Ford-Fulkerson Algorithm

- In the worst-case,
the Ford-Fulkerson algorithm takes $O(f \cdot (|V| + |E|))$ time.
 - Each flow augmentation takes $O(|V| + |E|)$ time to complete.
- The Ford-Fulkerson algorithm is not an efficient algorithm.
 - ***Its running time depends on the value of the input, which can be exponential in the length of the input.***
 - It is a pseudo-polynomial time algorithm.

Some Efficient Algorithms for Max-Flow

Efficient Algorithms for Max-Flow

- In the following,
we sketch a few efficient algorithms for max-flow and min-cut.
 - The capacity scaling algorithm, $O(|E|^2 \cdot \log f)$, where f is the maximum capacity of the edges.
 - The Edmonds-Karp algorithm, $O(|V| \cdot |E|^2)$.

It is beyond the scope of this course to inspect all the various flow algorithms. Refer to concluding remarks for further references.

The Capacity Scaling Algorithm

- The capacity scaling algorithm works as follows.
 - Let Δ be the maximum capacity of the edges.
 - While $\Delta > 0$, do
 - Repeatedly compute f -augmenting path with value at least Δ in G_f and augment f by Δ until there is none.
 - Divide Δ by 2.
 - Output f .

The Capacity Scaling Algorithm

- It can be shown (by induction) that,
 - In each iteration,
there are at most $O(|E|)$ f -augmenting paths with value $\geq \Delta$ in G_f .
- Hence, the total time complexity is $O(\log f \cdot |E|^2)$.
- Note that, in practice, $O(\log f) \ll O(|V|)$ almost always holds.

The Capacity Scaling Algorithm

- The capacity scaling algorithm is very easy to implement.
 - Almost as easy as the Ford-Fulkerson.
 - It takes less than 100 lines (with ample spacing and line-breaks) using only standard DFS.

The Edmonds-Karp Algorithm

- The Edmonds-Karp algorithm works as follows.
 - While there exists f -augmenting paths in G_f , do
 - Compute a shortest f -augmenting paths P , using BFS.
 - Use P to augment f by Δ_P .
 - Output f .

The Edmonds-Karp Algorithm

- It can be shown that,
 - The length of the shortest f -augmenting path between iterations is always nondecreasing and is at most $O(|V|)$.
 - The algorithm exhausts the capacity of at least one edge in each iteration.
 - The length of the shortest augmenting path will increase in $O(|E|)$ rounds.
- Hence, the total time complexity is $O(|V| \cdot |E|^2)$.

Concluding Notes

The Dinic's Algorithm

- The Dinic's algorithm is one of the best practical algorithm for max-flow.
 - It runs in $O(|V|^2 \cdot |E|)$.
 - It computes a maximal set of shortest augmenting paths in each iteration.
 - Similar to the Hopcroft-Karp algorithm for maximum bipartite matching.

Combining the Dinic's with the Capacity Scaling

- We can use Dinic's approach to compute a maximal set of f -augmenting paths with value Δ in G_f .
 - Then,
the capacity scaling algorithm runs in $O(|V| \cdot |E| \cdot \log f)$.

The Best Algorithm for Max-Flow

- There are major breakthroughs in the max-flow problem in recent years.
- The best algorithm (so far) is given by the following research paper, which solves the max-flow problem in “almost linear time.”

Chen, Kying, Liu, Peng, Gutenberg, Sachdeva,
“Maximum Flow and Minimum-Cost Flow in Almost-Linear Time,”
arXiv:2203.00671, 2022.

This giant monster paper has 110 pages!!

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Said by someone who had read it...

- It runs in $O(|E|^{1+o(1)})$ time.
The hidden constant, however, is very large.
- It involves several complicated dynamic data structures.