Combinatorial Mathematics

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Monday 18:30 – 20:20

Outline

- Double Counting
- Principle of Inclusion-Exclusion
- Miscellaneous Topics
  - The Density of 0-1 Matrices

The Double Counting Principle

If the elements of a set are *counted in two different ways*, the answers are the same.

Handshaking Lemma.

At a party,

the number of guests who shake hands an odd number of times is even.

- Consider the graph $G = (V, E)$ defined on the guests, where $(u, v) \in E$ if and only if guest u and guest v have shook hands.
  - For each $v \in V$, the degree of v, denoted $d(v)$, is the number of handshakes the guest v has made.
  - The number of edges, $|E|$, is the total number of handshakes.

  Then, we have

$$\sum_{v \in V} d(v) = 2 \cdot |E|.$$

$2 \cdot |E|$ is even.

Each edge is counted twice.

Hence, the number of vertices with odd degree must be even.

- Let F be a set family on a ground set X.
  - For any $x \in X$, define $d(x)$, the degree of x, to be the number of sets in F that contain x.
- The previous identity is a special case of the following general identity.

Proposition 1.7.

Let F be a family of subsets of some ground set X. Then

$$\sum_{x \in X} d(x) = \sum_{A \in F} |A| \ .$$

- Note that, the set family is a concept equivalent to hypergraphs, where
  - The elements are the vertices, and
  - The subsets in F are the hyperedges.

Proposition 1.7.

Let F be a family of subsets of some ground set X. Then

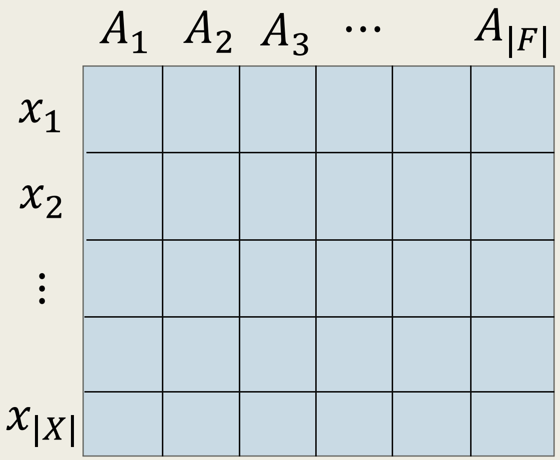
$$\sum_{x\in X} d(x) = \sum_{A\in F} |A| \ .$$

- Consider the $|X| \times |F|$ incidence matrix $M = \left(m_{x,A}\right)$, where

$$m_{x,A} = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{otherwise.} \end{cases}$$

- Then,
  - The $d(x)$ is the number of 1s in the x-th row.
  - $|A|$ is the number of 1s in the A-th column.

- ***The identity counts the number of 1s in the matrix M.***



The matrix M

Turán Number $T(n,k,l)$

- For any $l \leq k \leq n$, **the Turán number $\boldsymbol{T(n,k,l)}$** is

  the smallest number of l-element subsets

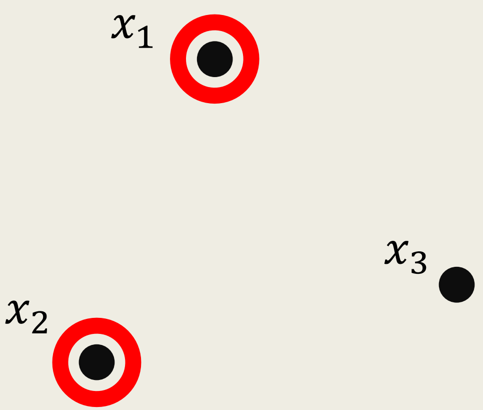
  of an n-element ground set X such that

  every k-element subset of X contains
  at least one of these l-element subsets.

Turán Number $T(n, k, l)$

- For any $n = 3$, $k = 2$, $l = 1$, we have

$$T(3,2,1) = 2\,.$$



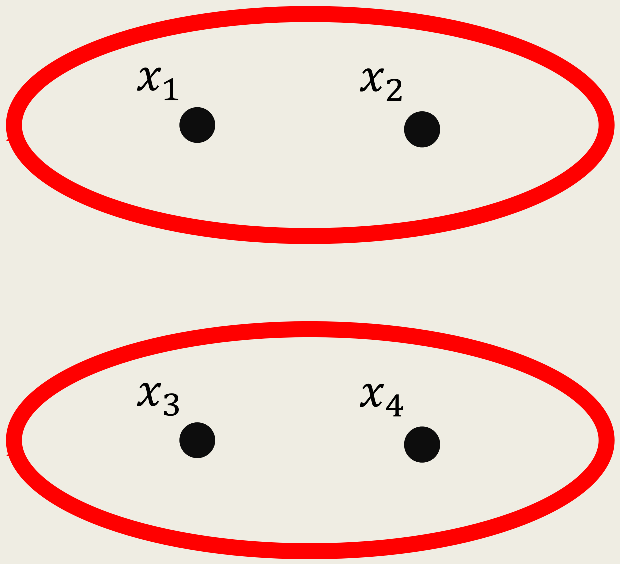
Any 2-element subset must contain $\{x_1\}$ or $\{x_2\}$.

It won’t suffice, if only one 1-element subset was chosen.

Turán Number $T(n,k,l)$

- For any $n = 4$, $k = 3$, $l = 2$, we have

$$T(4,3,2) = 2\,.$$



Any 3-element subset
must contain $\{x_1, x_2\}$ or $\{x_3, x_4\}$.

It won’t suffice,
if only one 2-element subset was chosen.

Proposition 1.9.

For all positive integers $l \leq k \leq n$,

$$T(n,k,l) \geq \binom{n}{l} \Big/ \binom{k}{l}.$$

- Let F be a smallest l-uniform family over X such that every k-element subset of X contains at least one member of F.

- Consider the $|F| \times \binom{n}{k}$ 0-1 matrix $M = (m_{A,B})$, where the rows are indexed by sets A in F and the columns are indexed by k-element subsets of X, and

$$m_{A,B} = \begin{cases} 1, & \text{if } A \subseteq B, \\ 0, & \text{otherwise.} \end{cases}$$

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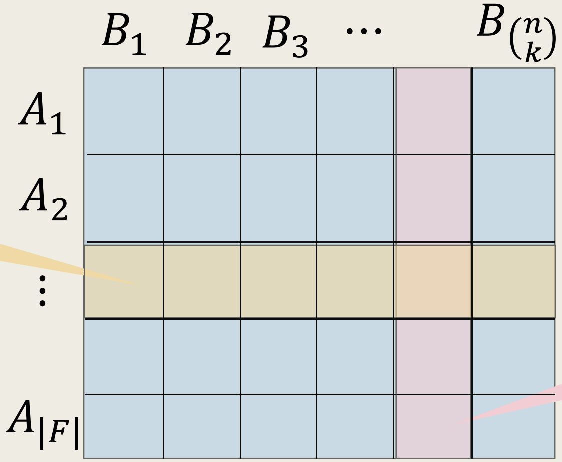
$$m_{A,B} = \begin{cases} 1, & \text{if } A \subseteq B, \\ 0, & \text{otherwise.} \end{cases}$$

For each l-element subset A, the number of k-element subsets containing the set A is exactly $\binom{n-l}{k-l}$.



The matrix M

Since every k-element subset of X contains at least one member of F, there exists at least one 1 in each column.

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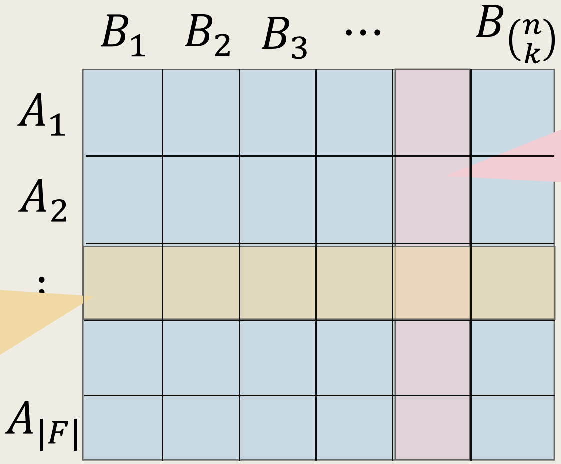

The matrix M

Since every k-element subset of X
contains at least one member of F,
there exists at least one 1
in each column.

- Let r_A be the number of 1s in row A and c_B the number of 1s in column B.
- Counting the number of 1s, we have

$$|F| \cdot \binom{n-l}{k-l} = \sum_{A \in F} r_A = \sum_{B} c_B \geq \binom{n}{k},$$

and

$$T(n,k,l) = |F| \geq \binom{n}{k} \Big/ \binom{n-l}{k-l} = \binom{n}{l} \Big/ \binom{k}{l}.$$

Average Number of Divisors

- How many numbers from $1, 2, \ldots, n$ divides at least one of the first n numbers, $1,2, \ldots, n$?
  - Let $t(n)$ be the number of divisors of n.

    We have $t(p) = 2$ for any prime number p, and $t(2^m) = m + 1$.

  - While $t(n)$ varies a lot for different choices of n, the average number of divisors,

$$\tau(n) := \frac{1}{n} \cdot \sum_{1 \le i \le n} t(i)$$

    is quite stable and is roughly $\ln n$ for all n.

Proposition 1.10.

$$|\,\tau(n) - \ln n\,| \leq 1\,.$$

- Consider the $n \times n$ 0-1 matrix $M = (m_{i,j})$, where $m_{i,j} = 1$ if and only i divides j.

	1	2	3	4	5	6	7	8	9
1	1	1	1	1	1	1	1	1	1
2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s in the i-th column is $t(i)$.

- Consider the $n \times n$ 0-1 matrix $M = (m_{i,j})$, where $m_{i,j} = 1$ if and only i divides j.

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2		1		1		1		1	
3			1			1			1
4				1				1	

The number of 1s in the i-th column is $t(i)$.

The number of 1s in the i-th row is $\lfloor n/i \rfloor$.

- Counting the number of 1s in the matrix, we have

$$\sum_{1 \le i \le n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1 \le i \le n} t(i) = n \cdot \tau(n).$$

- Counting the number of 1s in the matrix, we have

$$\sum_{1\le i\le n} \left\lfloor \frac{n}{i} \right\rfloor = \sum_{1\le i\le n} t(i) = n \cdot \tau(n) .$$

- Since we have $x - 1 \le \lfloor x \rfloor \le x$ for every real number x, we obtain

$$n \cdot \sum_{1\le i\le n} \frac{1}{i} - n \le n \cdot \tau(n) \le n \cdot \sum_{1\le i\le n} \frac{1}{i}$$

which implies that

$$H_n - 1 \le \tau(n) \le H_n ,$$

where $H_n := \sum_{1\le i\le n} \frac{1}{i} = \ln n + \gamma_n$ for some $0 \le \gamma_n \le 1$ is the n^{th}-harmonic number.

The Density of 0-1 Matrices

- Let H be an $m \times n$ 0-1 matrix and $0 \le \alpha \le 1$ be a real number.
  - We say that H is α-dense,
    if at least an α-fraction of all its entries are 1s.
  - Similarly, a row (column) is α-dense,
    if at least an α-fraction of its entries are 1s.

Lemma 2.13 (Grigni and Sipser 1995).

If H is 2α-dense, then either

1. There exists a row which is $\sqrt{\alpha}$-dense, or
2. At least $\sqrt{\alpha} \cdot m$ of the rows are α-dense.

Note that, $\sqrt{\alpha} \ge \alpha$ when $\alpha \le 1$.

Lemma 2.13 (Grigni and Sipser 1995).

If H is 2α-dense, then either

1. There exists a row which is $\sqrt{\alpha}$-dense, or
2. At least $\sqrt{\alpha} \cdot m$ of the rows are α-dense.

- Suppose that both of the cases do not hold.
  - By 2, less than $\sqrt{\alpha} \cdot m$ rows are α-dense.
  - By 1, each of the above rows has less than $\sqrt{\alpha} \cdot n$ 1s.
  - Hence, the total number of 1s in these α-dense rows is $< \sqrt{\alpha} \cdot \sqrt{\alpha} \cdot mn$
- At most m rows are not α-dense,
  - Hence, the total number of 1s in these rows is $< \alpha \cdot mn$
- The total number of 1s in H is strictly less than $2\alpha \cdot mn$, a contradiction.

Q: How many rows or columns of an α-dense matrix will be "***dense enough***?"

- <u>*Let's use a more general setting*</u> to answer the above question.
  - Let $A_1, A_2, \ldots, A_k$ be finite sets, and consider the Cartesian product
  $$A = A_1 \times A_2 \times \cdots \times A_k \,.$$
  - Let $H \subseteq A$ be a subset of interests.
    - For any $b \in A_i$, define the degree of b in H as
    $$d_H(b) := |\{a \in H : a_i = b\}| \,,$$
    i.e., the number of elements in H whose i^{th}-coordinate is b.

For $m \times n$ 0-1 matrix,
H is the set of coordinates of the entries that are 1.

$d_H(b)$ is the number of 1s in row (column) b.

<u>*To relate the two concepts*</u>,

for $m \times n$ 0-1 matrix, we have
$A_1 = \{1,2,\ldots,m\}$, $A_2 = \{1,2,\ldots,n\}$,
and
$A = \{\,(i,j) : 1 \le i \le m, 1 \le j \le n\,\}$
is the coordinates of the entries.

- Let $A_1, A_2, \dots, A_k$ be finite sets, and consider the Cartesian product $A = A_1 \times A_2 \times \cdots \times A_k$.
- Let $H \subseteq A$ be a subset of interests.
  - For any $b \in A_i$, define the degree of b in H as $d_H(b) \coloneqq |\{a \in H: a_i = b\}|$, i.e., the number of elements in H whose i^{th}-coordinate is b.
  - We say that a point $b \in A_i$ is popular in H, if

$$d_H(b) \geq \frac{1}{2k} \cdot \frac{|H|}{|A_i|} ,$$

i.e., $d_H(b)$ is at least $1/2k$ fraction of the average of the elements in A_i.

    - Let $P_i \subseteq A_i$ be the set of all popular elements in A_i, and let

$$P \coloneqq P_1 \times P_2 \times \cdots \times P_k .$$

Row (column) b is popular,
if its number of 1s (and thus density) is at least $1/2k$ the average of the rows (columns).

The entries formed by popular rows and columns.

- Let $A_1, A_2, \ldots, A_k$ be finite sets, and consider the Cartesian product

$$A = A_1 \times A_2 \times \cdots \times A_k \ .$$

- Let $H \subseteq A$ be a subset of interests and $d_H(b)$ be the degree of b in H.
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Lemma 2.14 (Håstad).

$|P| \ > \ |H|/2.$

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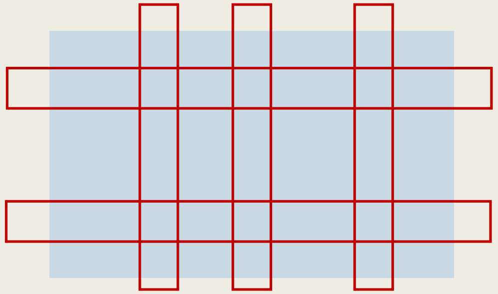
- We will prove that $|H \setminus P| < |H|/2$.
  - For every non-popular point $b \in A_i$, we have

$$d_H(b) = |\{a \in H : a_i = b\}| < \frac{1}{2k} \cdot \frac{|H|}{|A_i|}.$$

  - Counting the elements in $|H \setminus P|$, we have

$$|H \setminus P| \leq \sum_{1 \le i \le k} \sum_{b \notin P_i} d_H(b) < \sum_{1 \le i \le k} \sum_{b \notin P_i} \frac{1}{2k} \cdot \frac{|H|}{|A_i|}$$

$$\leq \sum_{1 \le i \le k} \frac{1}{2k} \cdot |H| = \frac{1}{2}|H|.$$



Any element in $|H \setminus P|$ is counted at least once.

$|A_i \setminus P_i| \leq |A_i|$

Q: How many rows or columns of an α-dense matrix will be "***dense enough***?"

- *<u>Let's use a more general setting</u>* to answer the above question.
  - Let $A_1, A_2, \dots, A_k$ be finite sets, and consider the Cartesian product

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  - Let $H \subseteq A$ be a subset of interests.
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For $m \times n$ 0-1 matrix,
H is the set of coordinates of the entries that are 1.

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<u>To relate the two concepts</u>,

for $m \times n$ 0-1 matrix, we have

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is the coordinates of the entries.

Q: How many rows or columns of an α-dense matrix will be "***dense enough***?"

- Interpret H as a subset of Cartesian product.
  - Lemma 2.14 says that, the size of $|P_1| \cdot |P_2|$ is lower-bounded by $|H|/2$.
  - Provided that $|H|$ is large, at least one of $|P_1|, |P_2|$ must be large.

Corollary 2.15.

In any 2α-dense 0-1 matrix H, either a $\sqrt{\alpha}$-fraction of its rows or a $\sqrt{\alpha}$-fraction of its columns (or both) are $\alpha/2$-dense.

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- Let P_1 is the set of rows with at least $\frac{1}{4} \cdot |H|/|A_1| \geq \alpha n/2$ ones, and

  P_2 is the set of column with at least $\frac{1}{4} \cdot |H|/|A_2| \geq \alpha m/2$ ones.

- By Lemma 2.14,

$$|P_1| \cdot |P_2| \geq \frac{1}{2} \cdot |H| \geq \alpha \cdot mn,$$

which implies that

$$\frac{|P_1|}{m} \cdot \frac{|P_2|}{n} \geq \alpha .$$

- Hence, either $\frac{|P_1|}{m} \geq \sqrt{\alpha}$ or $\frac{|P_2|}{n} \geq \sqrt{\alpha}$ must hold.

The Principle of Inclusion-Exclusion

- Let $A_1, A_2, \dots, A_n \subseteq X$ be given sets. For any $I \subseteq \{1,2,\dots,n\}$, define $A_I := \bigcap_{i \in I} A_i$ with the convention that $A_\phi = X$.

Theorem 3. (The inclusion-exclusion principle)

Let $A_1, A_2, \dots, A_n$ be a sequence of sets. We have

$$\left| \bigcup_{1 \le i \le n} A_i \right| = \sum_{\substack{I \subseteq \{1,2,\dots,n\}, \\ I \neq \emptyset}} (-1)^{|I|+1} \cdot |A_I|$$

$$= \sum_{0 < k \le n} \sum_{\substack{I \subseteq \{1,2,\dots,n\}, \\ |I| = k}} (-1)^{k+1} \cdot |A_I| .$$

Let's first derive $\left|\bigcap_{1 \le i \le n} \overline{A_i}\right|$.

Proposition 1.13 (Inclusion-Exclusion Principle).

Let $A_1, \ldots, A_n$ be subsets of X.
Then the number of elements of X which lie in none of the subsets A_i is

$$\sum_{I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|} \cdot |A_I| \ .$$

- Rewrite the sum as

$$\sum_{I} (-1)^{|I|} \cdot |A_I| \ = \ \sum_{I} \sum_{x \in A_I} (-1)^{|I|} \ = \ \sum_{x} \sum_{I : x \in A_I} (-1)^{|I|} \ .$$

- For each $x \in X$, consider the contribution of x to the above summation.
  - If $x \notin A_i$ for all i, then the only term in the sum to which x contributes is $I = \emptyset$, and the contribution is 1.

- Rewrite the sum as

$$\sum_{I}(-1)^{|I|}\cdot|A_I| = \sum_{I}\sum_{x\in A_I}(-1)^{|I|} = \sum_{x}\sum_{I:x\in A_I}(-1)^{|I|}.$$

- For each $x \in X$, consider the contribution of x to the above summation.
  - If $x \in A_i$ for all i, define

$$J = \{\, i : x \in A_i \,\} \neq \emptyset\,.$$

    Then $x \in A_I$ if and only if $I \subseteq J$.

  - Thus, the contribution is

$$\sum_{I\subseteq J}(-1)^{|I|} = \sum_{0\le i\le |J|}\binom{|J|}{i}\cdot(-1)^{i} = (1-1)^{|J|} = 0\,.$$

  - So, the overall sum is the number of points lying in none of the sets.

Proposition 1.14 (Inclusion-Exclusion Principle).

Let $A_1, \dots, A_n$ be subsets of X. Then

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n\}} (-1)^{|I|+1} \cdot |A_I| \ .$$

- We have $|A_1 \cup \cdots \cup A_n| = \left|A_\emptyset\right| - \left|\overline{A_1} \cap \cdots \cap \overline{A_n}\right|$.
- By Proposition 1.13, we obtain

$$|A_1 \cup \cdots \cup A_n| = \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n\}} (-1)^{|I|+1} \cdot |A_I| \ .$$