Combinatorial Mathematics

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Monday 18:30 – 20:20

Outline

- The Pigeonhole principle
  - The Erdős-Szekeres Theorem
  - The Dilworth Lemma for Posets
  - Mantel’s Theorem
  - Turán’s Theorem

The Pigeonhole Principle

(aka Dirichlet's principle)

If a set of size at least $rs + 1$ is partitioned into r sets, then some class receives at least $s + 1$ elements.

An "***integer version***" of the *Averaging Principle*

Proposition 1.

In any graph, there exist two vertices of the same degree.

- Let $G = (V, E)$ be a graph with $|V| = n$.
- The degree of any vertex is between 0 and $n - 1$.
  - If there is a vertex with degree 0, then there exists no vertex with degree $n - 1$, and vice versa.
  - There are at most $n - 1$ different values for the vertex degrees, while there are n vertices.
  - By the pigeonhole principle,
at least two vertices have the same degree.

Independent Set & Chromatic Number

- Let $G = (V, E)$ be a graph.
  - Let $\alpha(G)$ denote the maximum size of any independent set for G.
  - Let $\chi(G)$ denote the chromatic number of G,

    i.e., the minimum number of colors required to color V such that,

    no adjacent vertices have the same color.

  - Consider a coloring of V using $\chi(G)$ colors.

    Let $V_1, V_2, \dots, V_{\chi(G)}$ be the partition of the vertices by their colors.

    - For any $1 \leq i \leq \chi(G)$, the set V_i is an independent set for G.

Proposition 2.

In any graph G with n vertices, $n \leq \alpha(G) \cdot \chi(G)$.

- Proof 1.
  - Consider a coloring of V that uses $\chi(G)$ colors and $V_1, V_2, \ldots, V_{\chi(G)}$ be the partition of the vertices by their colors.
  - Since V_i is an independent set, $|V_i| \leq \alpha(G)$.
  - Hence, $n = \sum_i |V_i| \leq \alpha(G) \cdot \chi(G)$.

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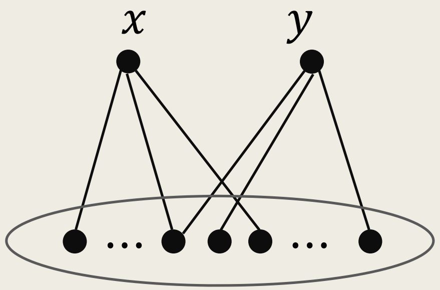
In any graph G with n vertices, $n \leq \alpha(G) \cdot \chi(G)$.

- Proof 2.
  - Consider a coloring of V that uses $\chi(G)$ colors and $V_1, V_2, \ldots, V_{\chi(G)}$ be the partition of the vertices by their colors.
  - By the pigeonhole principle, there exists some i with $|\mathrm{V}_i| \geq \frac{n}{\chi(G)}$.
  - Since V_i is an independent set, $\alpha(G) \geq |V_i|$.
  - By the above two inequalities, $n \leq \alpha(G) \cdot \chi(G)$.

Proposition 3.

Let G be a graph with n vertices. If every vertex has a degree of at least $(n-1)/2$, then G is connected.

- Proof.
  - We prove that, for any pair of vertices, say, x and y, either x and y are adjacent or have a common neighbor.
  - If x and y are not adjacent, then there are at least $n-1$ edges connecting them to the remaining vertices.
  - Since there are only $n-2$ other vertices, at least two of these $n-1$ edges connect to the same vertex.

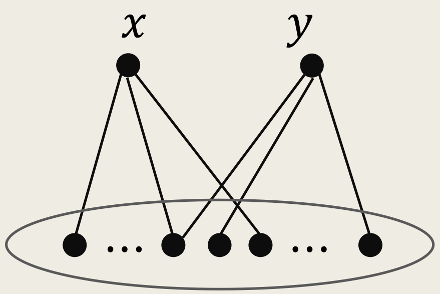


Some Remark.

- The statement from Proposition 3 is the best possible.
  - To see that, consider the graph that consists of two disjoint complete graphs, each of $n/2$ vertices.

    Then every vertex has degree $(n-2)/2$, and the graph is disconnected.

- Note that, what we actually proved is that, if every vertex has degree at least $(n-1)/2$, then the graph has diameter at most two.



The Erdős-Szekeres Theorem

Increasing / Decreasing Sequences

- Let $A = (a_1, a_2, \dots, a_n)$ be a sequence of n different numbers.
  - A subsequence of k terms of A is a sequence B of k distinct terms of A appearing in the same order in which they appear in A, i.e.,

    $B = \left(a_{i_1}, a_{i_2}, \dots, a_{i_k}\right)$, where $i_1 < i_2 < \cdots < i_k$.

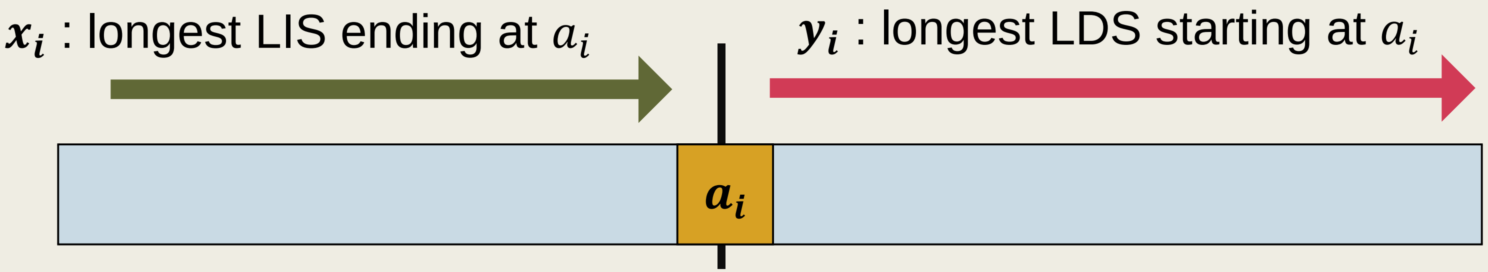
- A sequence is said to be increasing if $a_1 < a_2 < \cdots < a_n$ and decreasing if $a_1 > a_2 > \cdots > a_n$.

Theorem 5 (Erdős-Szekeres 1935).

Let $A = (a_1, a_2, \ldots, a_n)$ be a sequence of n different real numbers. If $n \geq sr + 1$, then either A has an increasing subsequence of length $s + 1$ or a decreasing subsequence of length $r + 1$.

- Proof. (due to Seidenberg 1959).

  For any $1 \leq i \leq n$, associate a_i with a pair (x_i, y_i), where
  - x_i is the length of the *longest increasing subsequence* ***ending at*** a_i.
  - y_i is the length of the *longest decreasing subsequence* ***starting at*** a_i.



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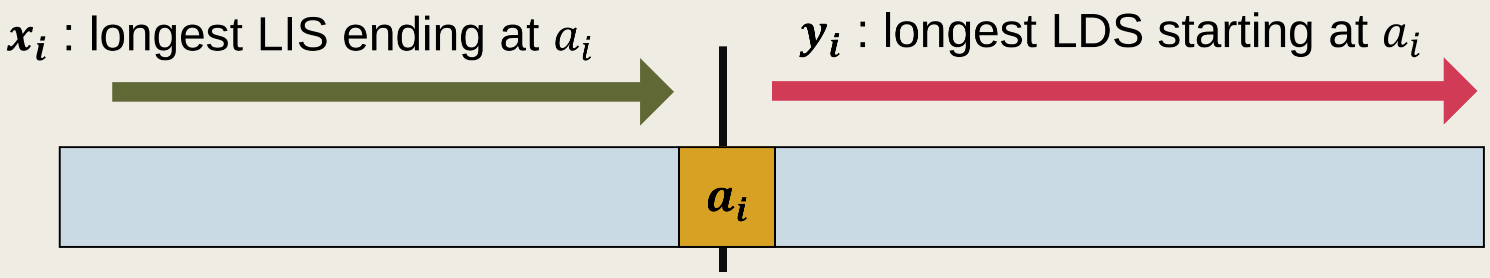
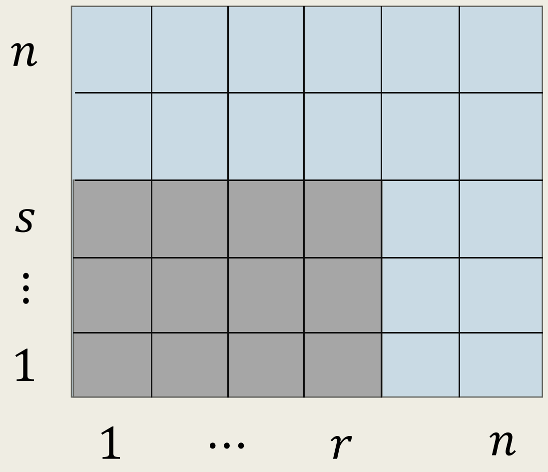


- For any $i \ne j$, say, $1 \le i < j \le n$, we have $(x_i, y_i) \ne (x_j, y_j)$.
  - If $a_i < a_j$, then $x_j \ge x_i + 1$.
  - If $a_i > a_j$, then $y_i \ge y_j + 1$.

The elements of the sequence are distinct, so one of the two conditions must hold.

- For any $i \neq j$, say, $1 \leq i < j \leq n$, we have $(x_i, y_i) \neq (x_j, y_j)$.
  - If $a_i < a_j$, then $x_j \geq x_i + 1$.
  - If $a_i > a_j$, then $y_i \geq y_j + 1$.



- Consider the $n \times n$ grids as pigeonholes.
  - By the above property, all the elements a_i correspond to a distinct grid.

- Consider the $s \times r$ submatrix.
  - Since $n > s \cdot r$, for some i, the element a_i corresponds to some grid outside the $s \times r$ submatrix.
  - Hence, either $x_i > s$ or $y_i > r$.

The Dilworth Lemma

for Partially Ordered Sets (Posets)

Partially Order Sets.

- A partial order on a set P is a binary relation $\preccurlyeq$ that is reflexive, antisymmetric, and transitive, i.e.,
  - *(reflexive)* $a \preccurlyeq a$, for all $a \in P$,
  - *(antisymmetric)* If $a \preccurlyeq b$ and $b \preccurlyeq a$, then $a = b$.
  - *(transitive)* If $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.
- Two elements $a, b \in P$ are said to be comparable if either $a \preccurlyeq b$ or $b \preccurlyeq a$.

Chain and Antichain.

- Let P be a set with partial order $\preccurlyeq$.
  - A subset $C \subseteq P$ is called a *chain*,
    if any pair of elements in C is comparable.
  - Dually, a subset $C \subseteq P$ is called an *antichain*,
    if all the pairs of elements in C are not comparable.

Chain and Antichain.

- For example,

  let $P = \{1, 2, 3, 4, 5, a, b, c, d\}$ and define the partial order $\leq$ as

  $$1 \leq 2 \leq 3 \leq 4 \leq 5, \text{ and}$$

  $$a \leq b \leq c \leq d.$$

  - Then, $\{4,2,3\}$ and $\{c, d\}$ are two chains,

    and $\{2, c\}$ is an antichain.

Lemma 6 (Dilworth 1950).

Let P be a set with a partial order $\preccurlyeq$.
If $|P| \geq sr + 1$, then there exists either a chain of size $s + 1$ or an antichain of size $r + 1$.

- Proof.
  - For any $a \in P$,
    let $\ell(a)$ denote the length of the longest chain ending at a.
  - Suppose that there exists no chain of size $s + 1$.
    - Then $\ell(a) \leq s$ for all $a \in P$.
    - We will show that, there exists an antichain of size $r + 1$.

- For any $a \in P$,
  let $\ell(a)$ denote the length of the longest chain ending at a.

- For $1 \leq i \leq s$, let A_i be the set of elements a with $\ell(a) = i$.

  - Then, $\boldsymbol{A_i}$ **must be an antichain**, for all $1 \leq i \leq s$.

    - Consider any $a, b \in A_i$ with $a \neq b$.
      By definition, we have $\ell(a) = \ell(b)$.

    - If a and b are comparable, say, $a \preccurlyeq b$,
      then, we add b to the longest chain ending at a.

      This gives a chain ending at a with size $\ell(b) + 1 = \ell(a) + 1$,
      a contradiction.

- Suppose that there exists no chain of size $s+1$.
  - Then $\ell(a) \leq s$ for all $a \in P$.

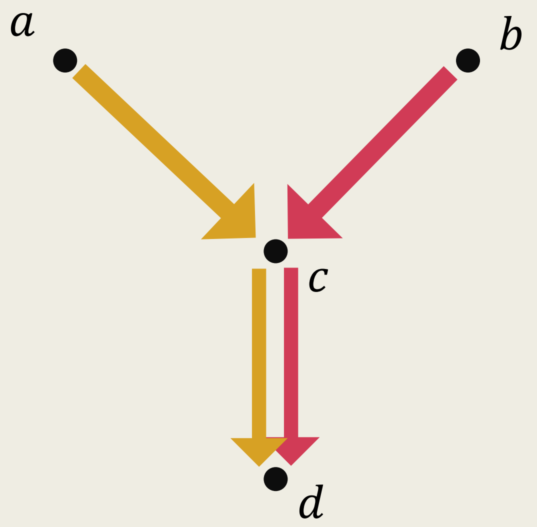
- For $1 \leq i \leq s$, let A_i be the set of elements a with $\ell(a) = i$.
  - Then, $\boldsymbol{A_i}$ **is an antichain**, for all $1 \leq i \leq s$.
  - $A_i \cap A_j = \emptyset$ for all $i \neq j$.
  - $A_1, A_2, \ldots, A_s$ forms a partition of P.

- Since $|P| \geq sr + 1$,

  by the pigeonhole principle, $|A_i| \geq r + 1$ for some i.

Some Note.

- The proof given in the textbook is wrong.
  - The greatest elements chosen in different maximal chains can be identical, and hence, comparable.



For example,

the two maximal chains, $\{a, c, d\}$ and $\{b, c, d\}$,

share the same greatest element d.

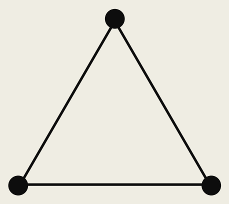
The Mantel’s Theorem

How many edges can a *triangle-free graph* have?

Alternatively,
how many edges can we add to a graph without creating a triangle?

The Maximum Number of Edges in a Triangle-free Graph.

- A triangle is a complete graph of 3 vertices.



- We know that, bipartite graphs do not contain any triangle.
  - So, $n^2/4$ edges are possible,
    achieved by complete bipartite graphs with two $n/2$ partite sets.
  - It turns out that, $n^2/4$ is also the best possible.

Theorem 7 (Mantel 1907).

If an n-vertex graph has more than $n^2/4$ edges, then it contains a triangle.

- Proof 1.
  - Let $G = (V, E)$ with $|V| = n$ and $|E| = m$.
  - Assume that G has no triangles. We will show that $m \leq n^2/4$.
    - Consider any $e = (x, y) \in E$.

      The pigeonhole principle guarantees that

$$d(x) + d(y) \leq n\ .$$

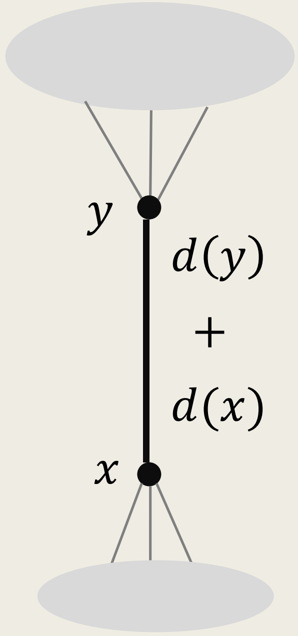
Otherwise, x and y share a common neighbor, and they jointly form a triangle.

- **Proof 1.**
  - Let $G=(V,E)$ with $|V|=n$ and $|E|=m>n^2/4$.
  - Assume that G has no triangles.
    - Consider any $e=(x,y)\in E$.

      The pigeonhole principle guarantees that

      $$d(x)+d(y)\le n\,.$$

Otherwise, x and y share a common neighbor, and they jointly form a triangle.



- Summing over all the edges, we obtain

$$\sum_{x\in V} d(x)^2 \quad = \boxed{\sum_{(x,y)\in E}\big(d(x)+d(y)\big) \ \le \ mn\,.}$$

By the double counting principle.

– We obtain

$$\sum_{x\in V} d(x)^2 \;\leq\; mn\,.$$

For any vector $u, v \in \mathbb{R}^n$,

$$|u \cdot v| \leq \|u\| \cdot \|v\|\,.$$

– Apply the ***Cauchy-Schwarz inequality*** to lower-bound $\sum_{x\in V} d(x)^2$.

Consider the two vectors $\begin{cases} u = (1,1,\ldots,1) \\ v = \big(d(v_1), d(v_2), \ldots, d(v_n)\big) \end{cases}$.

We have

$$|V| \cdot \sum_{x\in V} d(x)^2 \;\geq\; \left(\sum_{x\in V} d(x)\right)^2 \;=\; 4m^2\,.$$

By the double counting principle,

$\sum_{x\in V} d(x) = 2m$.

Hence, $m \leq n^2/4$.

Theorem 7 (Mantel 1907).

If an n-vertex graph has more than $n^2/4$ edges,
then it contains a triangle.

- Proof 2.
  - In the second proof, we count the number of edges using the property of the ***maximum independent set***.
  - Let $G = (V, E)$ with $|V| = n$.

    Assume that G has no triangles.
    - We will show that $|E| \leq n^2/4$.

- Let $G = (V, E)$ with $|V| = n$.

  Assume that G has no triangles.

If not, we get a triangle.

- Hence,

  for any $v \in V$, ***the neighbors*** of v ***form*** ***an independent set***.

- Let $A \subseteq V$ be the largest independent set in G.

  - None of vertex pairs in A is connected by an edge.

  - Hence, the set $B \coloneqq V \setminus A$ ***meets every edge of G***, and

$$|E| \leq \sum_{x \in B} d(x) \leq \sum_{x \in B} |A| = \boxed{|A| \cdot |B| \leq \left(\frac{|A| + |B|}{2}\right)^2} = n^2/4 \ .$$

Arithmetic and geometric mean inequality

Turán’s Theorem

How many edges can a K_ℓ*-free graph* have?

Alternatively,
how many edges can we add to a graph without creating a clique of size ℓ?

The Maximum Number of Edges in a K_ℓ-free Graph.

- A k-clique, denoted K_k, is a complete graph on k vertices.
- The Mantel's theorem states that, any K_3-free graph has at most $n^2/4$ edges.
  - What about k-cliques with $k > 3$?

Theorem 8 (Turán 1941).

If a graph $G = (V, E)$ with n vertices contains no $(k+1)$-cliques, where $k \geq 2$, then

$$|E| \leq \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2} \ .$$

- Proof.
  - We prove by induction on n.
  - The case with $n = 1$ is trivial,
    and the case $k = 2$ is proved by the Mantel's theorem.
  - Suppose that the inequality holds for graphs with at most $n - 1$ vertices.

- The case with $n = 1$ is trivial,
  and the case $k = 2$ is proved by the Mantel's theorem.

- Suppose that the inequality holds for graphs with at most $n - 1$ vertices.

- Let $G = (V, E)$ be an n-vertex graph with no $(k+1)$-cliques and with a maximal number of edges.

  - Adding any new edge to G will create a $(k+1)$-clique.

  - G must contain at least one k-clique.

    Let A be a k-clique in G, and let $B \coloneqq V \setminus A$.

  - Let $e_A, e_B, e_{A,B}$ denote the number of edges in A, in B, and that between A and B, respectively.

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  - Let A be a k-clique in G, and let $B \coloneqq V \setminus A$.
  - Let $e_A, e_B, e_{A,B}$ denote the number of edges in A, in B, and that between A and B, respectively.
  - We have $e_A = \binom{k}{2} = k(k-1)/2.$

    By the induction hypothesis, $e_B \le \left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2}.$

    Since G has no $(k+1)$-cliques,
    each $v \in B$ is adjacent to at most $k-1$ vertices in A.

    Hence, $e_{A,B} \le (k-1) \cdot (n-k).$

- Let $G = (V, E)$ be an n-vertex graph with no $(k+1)$-cliques and with a maximal number of edges.
  - Let A be a k-clique in G, and let $B \coloneqq V \setminus A$.
  - Let $e_A, e_B, e_{A,B}$ denote the number of edges in A, in B, and that between A and B, respectively.

- We obtain that

$$|E| = e_A + e_B + e_{A,B}$$

$$\leq \frac{k(k-1)}{2} + \left(1 - \frac{1}{k}\right) \cdot \frac{(n-k)^2}{2} + (k-1)(n-k)$$

$$= \left(1 - \frac{1}{k}\right) \cdot \frac{n^2}{2} .$$