Combinatorial Mathematics

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Monday 18:30 – 20:20

Outline

- Systems of Distinct Representatives
  - Hall's Matching / Marriage Theorem
  - The König-Egeváry Theorem

System of Distinct Representatives

Distinct Representative of Sets in a Family

- Let $F = \{S_1, S_2, \ldots, S_m\}$ be a set family.

- The elements $x_1, x_2, \ldots, x_m$ is called a set of distinct representatives for F, if the following two conditions hold.
  - $x_i \in S_i$ for all $1 \leq i \leq m$.
  - The elements $x_1, x_2, \ldots, x_m$ are distinct, i.e., $x_i \neq x_j$ for all $i \neq j$.

An Equivalent Formulation

- Let $G = (A, B, E)$ be a bipartite graph with partite sets A and B.

- A set of edges $M \subseteq E$ is called a matching,
  if all the endpoints of the edges in M are distinct.

  – We say that a vertex $v \in A \cup B$ is matched by M,
    if it is incident to some edge in M.

  – A matching M is called a matching for A,
    if it matches all the vertices in A.

Existence of Distinct Representative

- A natural question is,

  When do we have a set of distinct representatives?

- Alternatively,

  When can we be sure that, *a matching for a partite set exists*?

Hall’s Matching Condition

The necessary and sufficient condition for a matching to exist.

Theorem 5.1 (Hall's Theorem).

The set family $S_1, S_2, \ldots, S_m$ has a set of distinct representatives ***if and only if***

$$\left|\bigcup_{i\in I} S_i\right| \geq |I| \quad \text{for all } I \subseteq \{1,2,\ldots,m\}. \qquad (*)$$

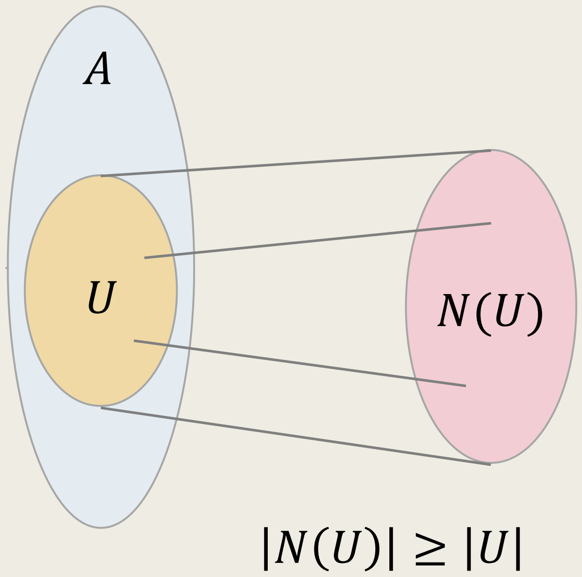
- It is clear that,
  $(*)$ is a necessary condition for distinct representatives to exist.
  - Surprisingly, the obvious necessary condition is also sufficient.

Some Remarks.

- For bipartite graph $G = (A, B, E)$,
  Hall's theorem translates to the following.
  - There is a matching for A if and only if

$$|N(U)| \geq |U|, \text{ for all } U \subseteq A.$$

i.e., for any $U \subseteq A$, there is always a sufficient number of candidates to be matched to in B.



Theorem 5.1 (Hall's Theorem).

The set family $S_1, S_2, \dots, S_m$ has a set of distinct representatives ***if and only if***

$$\left|\bigcup_{i \in I} S_i\right| \geq |I| \quad \text{for all } I \subseteq \{1,2,\dots,m\}. \qquad (*)$$

- Proof.
  - The direction ($\Longrightarrow$) is clear.
  - We prove the direction ($\Longleftarrow$) by induction on m.
    The case $m = 1$ holds trivially.
  - Assume that the statement holds for families with fewer than m sets.

- Proof. (continue)
  - Assume that the statement holds for families with fewer than m sets.

    We have the following two cases.

    1. For each k, where $1 \leq k < m$,
       the union of any k sets contains more than k elements.

    2. For some k, where $1 \leq k < m$,
       the union of some k sets contains exactly k elements.

There are *always*
more (candidates) than we need.

For some combination,
the number of candidates is tight.

- We construct the set of representatives as follows.

1. For each k, where $1 \leq k < m$,
   the union of any k sets ***contains more than k elements***.

   - Pick an arbitrary $x \in S_1$ to be the representative for S_1.

     Remove x from all the remaining $m-1$ sets.

   - Then, the union of any k remaining sets, where $1 \leq k \leq m-1$,
     still contains at least k elements.

   - By the induction hypothesis, there exist distinct representatives, other than x, for the remaining sets.

     Together we have a set of distinct representatives for $S_1, S_2, \ldots, S_m$.

We remove *at most one element* from each set.

2. For some k, where $1 \leq k < m$,
the union of some k sets contains exactly k elements.

The condition holds initially.

We remove a minimum number of candidates .

So, there's still an adequate number of candidates left.

- By the induction hypothesis,
  there exist k distinct representatives for these sets.

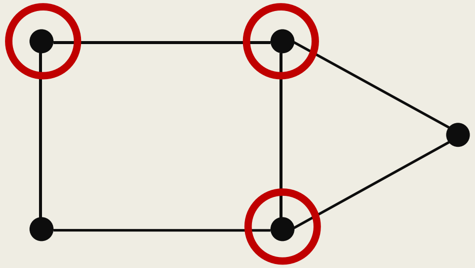
  Remove the k elements from the remaining $m - k$ sets.

- Then, the union of any s remaining sets, where $1 \leq s \leq m - \mathrm{k}$,
  must contains at least s elements.
  - If not, the union of these s sets with the above k sets contains less than $s + k$ elements, a contradiction.
- By induction hypothesis, there exist distinct representatives for these remaining $m - k$ sets.

The König-Egeváry Min-Max Theorem

In bipartite graphs, the size of the ***maximum matching*** is equal to the size of the ***minimum vertex cover***.

Vertex Cover of a Graph.



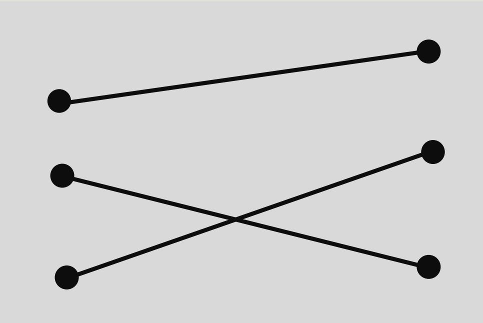
- Let $G = (V, E)$ be a graph.
- A ***vertex cover*** of G is a subset $U \subseteq V$ of vertices such that, any edge $e \in E$ has at least one endpoint in U.
  - Intuitively, we use the vertices in U to cover the edges in E.
- We want to select as few vertices as possible to cover the edges in the graph.

Theorem 5.5 (König-Egeváry 1931).

In a bipartite graph, the size of ***maximum matching*** is equal to the size of ***minimum vertex cover***.

Proof.

- Let $G = (U, V, E)$ be a bipartite graph.
  - Let M and C be a maximum matching and a minimum vertex cover for G, respectively.
  - It is clear that $|C| \geq |M|$.
    - The endpoints of the edges in M are distinct.
    - It takes *at least one vertex* to cover *each edge in M*.

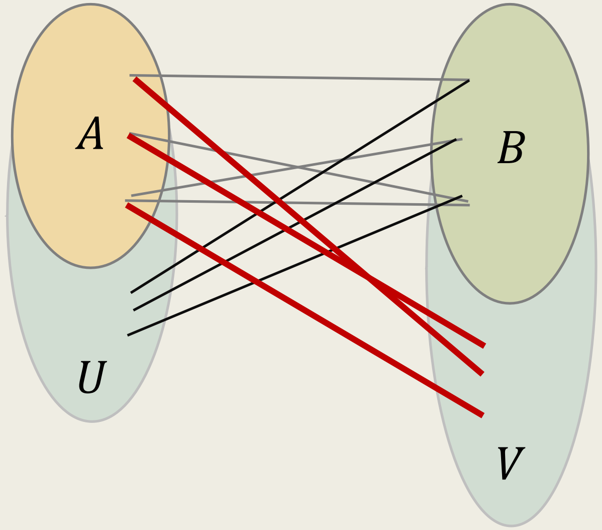


The matching M

<u>Theorem 5.5 (König-Egeváry 1931).</u>

In a bipartite graph, the size of ***maximum matching*** is equal to the size of ***minimum vertex cover***.

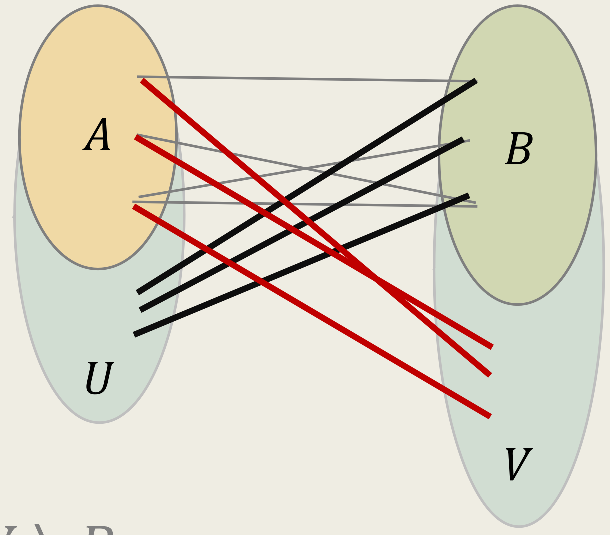
Proof.



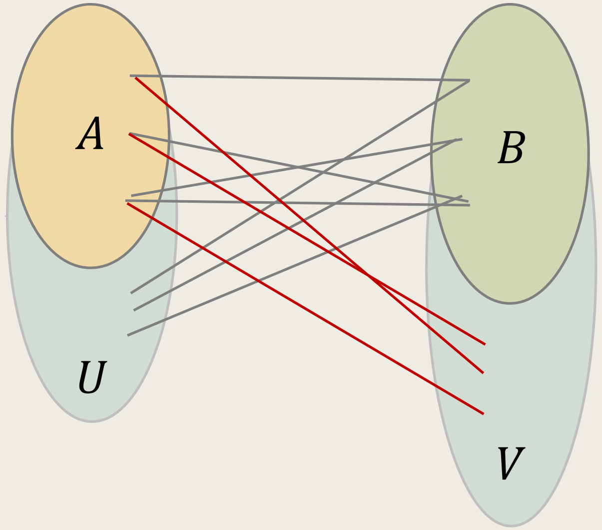
- It suffices to prove that $|M| \geq |C|$.
  - Let $A \coloneqq U \cap C$ and $B \coloneqq V \cap C$.
  - We will prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.

Proof.

- It suffices to prove that $|M| \geq |C|$.
  - Let $A \coloneqq U \cap C$ and $B \coloneqq V \cap C$.
  - We will prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.
  - **If the above is true**, then by a similar argument, there exists a matching M_B for B to $U \setminus A$.
  - The endpoints of the edges in $M_A \cup M_B$ are distinct.
    - So, $M_A \cup M_B$ is a matching of size $|A| + |B| = |C|$.
    - Hence, $|M| \geq |C|$.



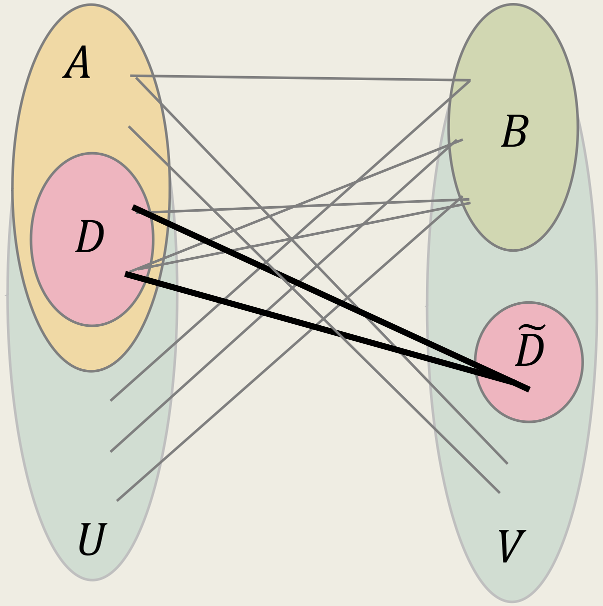
It suffices to prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.



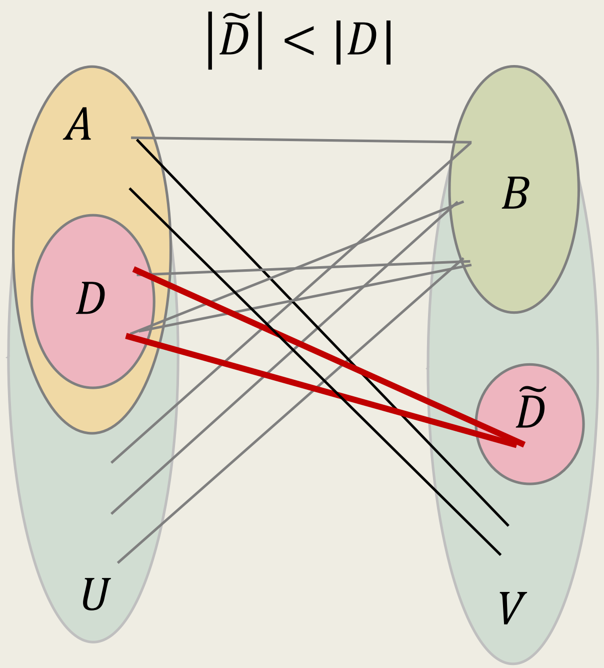
- Suppose that there exists no such matching.
  - Then, by Hall's matching theorem, there exists some $D \subseteq A$, such that

$$| \; N(D) \; \cap \; (V \setminus B) \; | \; < \; |D| \, .$$

  - Indeed, if $|N(D) \cap (V \setminus B)\,| \geq |D|$ holds for all $D \subseteq A$, then there exists a matching from A to $V \setminus B$.
  - Since there is no such matching, there must be such a $D \subseteq A$ with $|\,N(D) \cap (V \setminus B)\,| < |D|$.



It suffices to prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.



- If not, there exists some $D \subseteq A$, such that

  $$| \; N(D) \; \cap \; (V \setminus B) \; | \; < \; |D| \; .$$

  - Let $\widetilde{D} \coloneqq N(D) \cap (V \setminus B)$, then $|\widetilde{D}| < |D|$.
  - Observe that, $\left((A \setminus D) \cup \widetilde{D}\right) \cup B$ is a valid vertex cover for G.
    - There are four categories of edges in G.
      - $E_{A,B}$, $E_{U \setminus A,B}$: covered by B.
      - $E_{A \setminus D, V \setminus B}$: covered by $A \setminus D$.
      - $E_{D,\widetilde{D}}$: covered by $\widetilde{D}$.

D is replaceable by $\widetilde{D}$.

Since $C = A \cup B$ is a vertex cover,
there is not edge between $U \setminus A$ and $V \setminus B$.

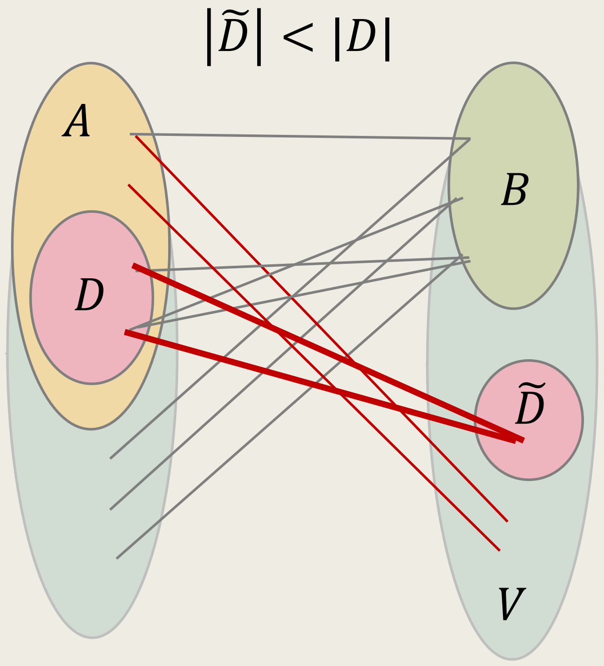
It suffices to prove that, there exists a matching M_A that matches all the vertices in A to the vertices in $V \setminus B$.

- If not, there exists some $D \subseteq A$, such that

$$| \ N(D) \ \cap \ (V \setminus B) \ | \ < \ |D| \ .$$

  - Let $\widetilde{D} := N(D) \cap (V \setminus B)$, then $|\widetilde{D}| < |D|$.
  - Then, $\left((A \setminus D) \cup \widetilde{D}\right) \cup B$ is a valid vertex cover with size strictly smaller than $C = A \cup B$,

    a contradiction.



D is replaceable by $\widetilde{D}$.