Problem 1 (20%). How many integer solutions are there to $x_1 + x_2 + x_3 + x_4 + x_5 = 31$ with

1. $x_i \geq 0$.
2. $x_i > 0$.
3. $x_i > i$ for all $i = 1, 2, 3, 4, 5$.

Problem 2 (30%). Prove the following identities using *either* the path-walking argument or the committee selection argument.

1. For any $n, r \in \mathbb{Z}^{\geq 0}$,
$$\sum_{0 \leq k \leq r} \binom{n+k}{k} \;=\; \binom{n+r+1}{r}.$$
2. For any $m, n, r \in \mathbb{Z}^{\geq 0}$ with $0 \leq r \leq m+n$,
$$\sum_{0 \leq k \leq r} \binom{m}{k}\binom{n}{r-k} \;=\; \binom{m+n}{r}.$$

Problem 3 (20%). Use the principle of Inclusion-Exclusion to calculate the number of positive integers that are ≤ 70 and relatively prime to 70.

Problem 4 (30%). Let $\mathcal{F}$ be a family of subsets of an n-element set X with the property that any two members of $\mathcal{F}$ meet, i.e., $A \cap B \neq \emptyset$ for all $A, B \in \mathcal{F}$. Suppose that for any proper subset A of X that is not in $\mathcal{F}$, A does not meet all of the members of $\mathcal{F}$, i.e., for any $A \subset X$, $A \notin \mathcal{F}$, there exists a set $B \in \mathcal{F}$ such that $A \cap B = \emptyset$. Prove that

$$|\mathcal{F}| \;=\; 2^{n-1}.$$

Hint: Consider the sets and their complements. Establish "$\leq$" and "$\geq$" separately.