Problem 1 (20%)**.** Let X, Y be random variables. The variance of a random variable X is defined as $\mathrm{Var}[X] := \mathrm{E}\left[(X - \mathrm{E}[X])^2\right]$. Prove that

1. $\mathrm{E}[aX + bY] = a \cdot \mathrm{E}[X] + b \cdot \mathrm{E}[Y]$ for any constant a, b.
2. If X and Y are independent, then $\mathrm{E}[X \cdot Y] = \mathrm{E}[X] \cdot \mathrm{E}[Y]$ and
   $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
3. $\mathrm{Var}[X] = \mathrm{E}[X^2] - \mathrm{E}[X]^2$. *Hint:* Use the fact that $\mathrm{E}[X \cdot \mathrm{E}[X]] = \mathrm{E}[X]^2$.

Problem 2 (20%)**.** Consider the slides of Week 2.
Prove that the graphs H_i defined in the proof of Theorem 3 are bicliques.

Problem 3 (20%)**.** Let $\mathrm{bc}(G)$ denote the smallest weight of a biclique covering of a graph G. Show that, if an n-vertex graph G has no independent set of size larger than α, then $\mathrm{bc}(G) \geq n \log_2(n/\alpha)$.

Hint: Prove as in the proof of the lower bound in Theorem 3. Show that $\mathrm{E}[X] \leq \alpha$.

Problem 4 (20%)**.** Let $\mathcal{F}$ be a family of subsets, where

$$|A| \geq 3 \text{ for any } A \in \mathcal{F} \qquad \text{and} \qquad |A \cap B| = 1 \text{ for any } A, B \in \mathcal{F},\ A \neq B.$$

Suppose that $\mathcal{F}$ is not 2-colorable. Let x, y be any elements that appear in $\mathcal{F}$, i.e., $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$ for some $A, B \in \mathcal{F}$. Prove that:

- (i) x belongs to at least two members of $\mathcal{F}$.
- (ii) There exists some $C \in \mathcal{F}$ such that $\{x, y\} \subseteq C$.

Hint: Construct proper coloring to prove the properties. For (i), consider a particular A with $x \in A \in \mathcal{F}$. Color $A \setminus \{x\}$ red and the remaining blue. Show that this leads to the conclusion of (i). For (ii), consider particular A, B with $x \in A \in \mathcal{F}$ and $y \in B \in \mathcal{F}$. Color $(A \cup B) \setminus \{x, y\}$ red and the remaining blue. Prove that it leads to (ii).

Problem 5 (20%)**.** Let $G = (A \cup B, E)$ be a bipartite graph, d be the minimum degree of vertices in A and D the maximum degree of vertices in B. Assume that $|A|d \geq |B|D$.

Show that, for every subset $A_0 \subseteq A$ with the density α defined as $\alpha := |A_0|/|A|$, there exists a subset $B_0 \subseteq B$ such that:

1. $|B_0| \geq \alpha \cdot |B|/2$,
2. every vertex of B_0 has at least $\alpha D/2$ neighbors in A_0, and
3. at least half of the edges leaving A_0 go to B_0.

Hint: Let B_0 consist of all vertices in B that have at least $\alpha D/2$ neighbors in A_0. First prove (3) and then (1).