Problem 1 (20%). How many integer solutions of $x_1 + x_2 + x_3 + x_4 = 28$ are there with

1. $0 \le x_i \le 12$?

2. $-10 \le x_i \le 20$?

3. $x_i \ge 0, \quad x_1 \le 6, \quad x_2 \le 10, \quad x_3 \le 15, \quad x_4 \le 21$?

Problem 2 (30%). Let $\mathcal{F}$ be a set family for the ground set X and $d(x)$ be the degree of any $x \in X$. Use the double counting principle to prove the following identities.

$$\sum_{x \in Y} d(x) \;=\; \sum_{A \in \mathcal{F}} |Y \cap A| \;\text{ for any } Y \subseteq X.$$

$$\sum_{x \in X} d(x)^2 \;=\; \sum_{A \in \mathcal{F}} \sum_{x \in A} d(x) \;=\; \sum_{A \in \mathcal{F}} \sum_{B \in \mathcal{F}} |A \cap B|.$$

Problem 3 (20%). Prove that for any two sets $I \subseteq J$,

$$\sum_{I \subseteq K \subseteq J} (-1)^{|K \setminus I|} = \begin{cases} 1, & \text{if } I = J, \\ 0, & \text{if } I \neq J. \end{cases}$$

Hint: Rewrite the summation and use the binomial theorem.

Problem 4 (15%). Let H be a 2α-dense 0-1 matrix. Prove that at least an $\alpha/(1-\alpha)$ fraction of its rows must be α-dense.

Problem 5 (15%). Let X be a finite set and $A_1, A_2, \ldots, A_m$ be a partition of X into mutually disjoint blocks. Given a subset $Y \subseteq X$, consider the partition $Y = B_1 \cup B_2 \cup \cdots \cup B_m$ with the blocks B_i defined as $B_i := A_i \cap Y$. For any $1 \le i \le m$, we say that the block B_i is λ-*large* if

$$\frac{|B_i|}{|A_i|} \;\ge\; \lambda \cdot \frac{|Y|}{|X|}.$$

Show that, for every $\lambda > 0$, at least $(1-\lambda) \cdot |Y|$ elements of Y belong to λ-large blocks.