Problem 1 (25%). Show that, for any positive integer n, there is a multiple of n that contains only the digits 7 or 0.

Hint: Consider all the numbers a_i of the form $77\ldots7$, with i sevens, for $i = 1, 2, \ldots n+1$, and the value a_i modulo n.

Problem 2 (25%). Suppose that M, M' are matchings in a bipartite graph G with partite sets A and B. Suppose that, the matching M matches the vertices S for some $S \subseteq A$ and the matching M' matches the vertices T for some $T \subseteq B$. Prove that there is a matching that matches all the vertices of $S \cup T$.

Hint: Consider $M \cup M'$. Show that it is a union of paths and cycles with even length, and then obtain a matching for $S \cup T$ from $M \cup M'$.

Problem 3 (20%). Prove the Corollary 5.2 in the lecture note #9. The arguments in the lecture note is not clear enough. Rewrite the proof argument clearly in your own way.

Problem 4 (15%). Consider a group of m girls and n boys, where we draw an edge between a girl and a boy if they know each other to form a bipartite graph. Show that, there exist a subset of t boys who can be matched, i.e., there exists a matching between these t boys and the girls, if and only if any subset of boys, say, K, know at least $|K| + t - n$ of the girls.

Hint: Add additional $n - t$ popular girls who are known to all the boys. Show that at least t boys can be matched in the original problem if and only if all the boys can be matched in the new problem. Apply the Hall's matching theorem to the new problem.

Problem 5 (15%). Let $\alpha(G)$ be the *independence number* of a graph G, i.e., the maximum size of any independent set in G. Prove the following dual version of Turán's theorem:

> If G is a graph with n vertices and $nk/2$ edges, where $k \geq 1$, then we have
>
> $$\alpha(G) \geq n/(k+1).$$