**Problem 1** (30%). Prove that every set of $n+1$ distinct integers chosen from $\{1, 2, \ldots, 2n\}$ contains a pair of consecutive numbers and a pair whose sum is $2n + 1$.

For each $n$, exhibit two sets of size $n$ to show that the above results are the best possible, i.e., sets of size $n + 1$ are necessary.

*Hint:* Use pigeonholes $(2i, 2i - 1)$ and $(i, 2n - i + 1)$ for $1 \leq i \leq n$.

**Problem 2** (30%). Let $G = (V, E)$ be a graph. Denote by $\chi(G)$ the minimum number of colors needed to color the vertices in $V$ such that, no adjacent vertices are colored the same.

Prove that, $\chi(G) \leq \Delta(G) + 1$, where $\Delta(G)$ is the maximum degree of the vertices.

*Hint:* Order the vertices $v_1, v_2, \ldots, v_n$ and use greedy coloring. Show that it is possible to color the graph using $\Delta(G) + 1$ colors.

**Problem 3** (20%). Use nonnegative edge weights and construct a 4-vertex edged-weighted graph in which the maximum-weight matching is not a maximum-cardinality matching.

*Note:* The cardinality is referred to the size of a set.

**Problem 4** (20%). Let $\mathcal{F}$ be a family of sets, where $|F| \geq 2$ for all $F \in \mathcal{F}$. Let $A, B$ be two sets with $|A| = |B|$, and both $A$ and $B$ intersect all the members of $\mathcal{F}$. Furthermore, no sets with smaller size than $A$ can intersect all the members of $\mathcal{F}$, i.e., for all set $C$ with $|C| < |A|$, $C$ does not intersect all the members of $\mathcal{F}$.

Consider a bipartite set $G$ with partite sets $A$ and $B$, where there is an edge between $a \in A$ and $b \in B$ if there exists some $F \in \mathcal{F}$ such that $a \in F$ and $b \in F$. Prove that $G$ has a perfect matching.

*Hint:* For any $I \subseteq A$, let $S(I) \subseteq B$ be the neighbors of $I$ in $G$. Show that the set $A' := (A \setminus I) \cup S(I)$ intersect all the members of $\mathcal{F}$.