1 The Maximum Matching Problem

Definition 1 (The Maximum Matching Problem)**.** Given a graph $G = (V, E)$, compute a matching $M \subseteq E$ that has the maximum cardinality.

A generic algorithm for this problem goes as follows.

1. $M \longleftarrow \emptyset$.
2. Repeatedly compute an M-augmenting path P in G until there exists none.
   - Update M by setting $M \longleftarrow M \triangle P$.
3. Output M.

The correctness of the algorithm is based on the Berge's theorem. Note that, $M \triangle P := (M \setminus P) \cup (P \setminus M)$ is the symmetric difference between M and P.

1.1 The Augmenting Path Problem in Bipartite Graphs

Definition 2 (The Augmenting Path Problem)**.** Given a graph $G = (V, E)$ and a matching $M \subseteq E$, determine if there exists an M-augmenting path and compute one if it exists.

The Augmenting Path Problem in bipartite graphs can be answered in $O(n + m)$ time.

For any $v \in V$, let $\ell(v)$ denote the vertex to which v is matched by M. $\ell(v)$ is defined to be -1 if v is unmatched.

1. Mark all the vertices as *unvisited*.
2. For each *unmatched* vertex $u \in V$, do
   - If Aug-Path(u) returns true,
     then report "Yes".
3. Report "No".

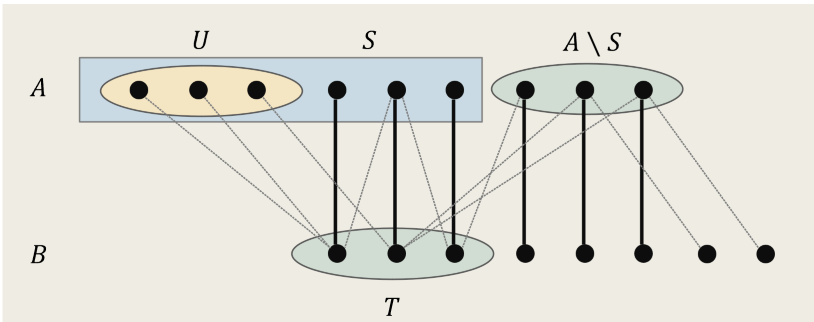
The recursive procedure Aug-Path(u) goes as follows.

1. Mark u as *visited*
2. For each neighbor v of u, do
   - If v is *unmatched*, or, if $\ell(v)$ is *unvisited* and Aug-Path($\ell(v)$) returns true, then
     - Match u with v. // set $\ell(u) = v$, $\ell(v) = u$.
     - Return true.
3. Return false.

Note that, when the Augmenting Path algorithm reports "Yes," the corresponding M-augmenting path is given by the recursive call Aug-Path(u) that results in "Yes."

Define the following notations.

- Let A, B be the two partite sets of G.
- Let U be the set of unmatched vertices in A.
- Let S the set of vertices in A that are marked as visited.
- Let T be the set of vertices that are matched to $S \setminus U$ by M.



Then, when the Augmenting Path algorithm reports "No," the set $C := (A \setminus S) \cup T$ is a vertex cover for G with size M.

1.2 The Maximum Matching Problem in Bipartite Graphs

The Maximum Matching Problem in bipartite graphs can be solved in $O(\sqrt{n} \cdot m)$ time. The following is an $O(nm)$ time algorithm.

- Set $\ell(v) = -1$ for all $v \in V$.
- Repeatedly apply the Augmenting Path algorithm to enlarge the matched pairs until it reports "No."
- Output the matched pairs.